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The Declining Price Anomaly Is Not Universal in Multi-Buyer Sequential Auctions (but almost is)

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Abstract

The declining price anomaly states that the price weakly decreases when multiple copies of an item are sold sequentially over time. The anomaly has been observed in a plethora of practical applications. On the theoretical side, Gale and Stegeman (*Games and Economic Behavior*, **36**(1), 74–103, 2001) proved that the anomaly is guaranteed to hold in full-information sequential auctions with exactly two buyers when one item is sold in each time period. We prove that the declining price anomaly is *not* guaranteed in full-information sequential auctions with three or more buyers. This result applies to both first-price and second-price sequential auctions. Moreover, it applies regardless of the tie-breaking rule used to generate equilibria in these sequential auctions. To prove this result we provide a refined treatment of subgame perfect equilibria that survive the iterative deletion of weakly dominated strategies and use this framework to experimentally generate a very large number of random sequential auction instances. In particular, our experiments produce an instance with three bidders and eight items that, for a specific tie-breaking rule, induces a non-monotonic price trajectory. Theoretical analyses are then applied to show that this instance can be used to prove that for every possible tie-breaking rule there is a sequential auction on which it induces a non-monotonic price trajectory. On the other hand, our experiments show that non-monotonic price trajectories are extremely rare. In over eighteen million experiments only a 0.000183 proportion of the instances violated the declining price anomaly.

Keywords Declining price anomaly · Sequential auctions

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1 Introduction

In a sequential auction identical copies of an item are sold over time. In a private values model with *unit-demand*, risk neutral buyers, Milgrom and Weber [23, 32] showed that the sequence of prices forms a martingale. In particular, expected prices are constant over time.[1] In contrast, on attending a wine auction, [2] made the surprising observation that prices for identical lots declined over time: "The law of the one price was repealed and no one even seemed to notice!" This *declining price anomaly* was also noted in sequential auctions for the disparate examples of livestock [8], Picasso prints [25] and satellite transponder leases [23]. Indeed, the possibility of decreasing prices in a sequential auction was raised by [28] nearly sixty years ago. An assortment of reasons have been given to explain this anomaly. In the case of wine auctions, proposed causes include absentee buyers utilizing non-optimal bidding strategies [14] and the *buyer's option rule* where the auctioneer may allow the buyer of the first lot to make additional purchases at the same price [7]. Minor non-homogeneities amongst the items can also lead to falling prices. For example, in the case of art prints the items may suffer slight imperfections or wear-and-tear; as a consequence, the auctioneer may sell the prints in decreasing order of quality [25]. More generally, a decreasing price trajectory may arise due to risk-aversion, such as non-decreasing, absolute risk-aversion [21] or aversion to price-risk ([22]; see also [16]). Further potential economic and behavioural explanations have been provided in [4, 14, 30].

Of course, most of these explanations are context-specific. However, the declining price anomaly appears more universal. In fact, in practice the anomaly is ubiquitous: It has now been observed in sequential auctions for antiques [15], commercial real estate [19], condominiums [3], fish [13], flowers [31], fur [18], lobsters [27], jewellery [10], paintings [6], stamps [29] and wool [9].

Given the plethora of examples, the question arises as whether this property is actually an anomaly. Declining prices are indeed an anomaly, even if empirically common, for multiple reasons. In unpublished work, Milgrom and Weber showed that with single-unit demand bidders, expected prices should *increase* rather than decrease over time due to information release in earlier rounds. Additionally, [2] observed that despite the fact that declining prices are common knowledge among participants at wine auctions, even single-unit buyers often choose to buy at the higher price rather than save money by waiting for later rounds. In groundbreaking work on the theoretical side, [12] proved that this pattern is *not* an anomaly in the equilibria of sequential auctions with *two* bidders. Specifically, in *second-price* sequential auctions with two multiunit-demand buyers, prices are weakly decreasing over time at the unique subgame perfect equilibrium that survives the iterative deletion of weakly dominated strategies. Moreover, this result applies regardless of the valuation functions of the buyers; the result also extends to the corresponding equilibrium in *first-price* sequential auctions. It is worth highlighting here two important aspects of the model studied in [12]. First, Gale and Stegeman consider

[1] If the values are affiliated then prices can have an upwards drift.

multiunit-demand buyers whereas prior theoretical work had focussed on the simpler setting of unit-demand buyers. As well as being of more practical relevance (see the many examples above), multiunit-demand buyers can implement more sophisticated bidding strategies. Therefore, it is not unreasonable that equilibria in the multiunit-demand setting may possess more interesting properties than equilibria in the unit-demand setting. Second, they study an auction with *full information*. The restriction to full information is extremely useful here as it separates away informational aspects. Hence, it allows one to focus on the strategic properties caused purely by the sequential sales of items and not by a lack of information.

1.1 Results and Overview of the Paper

The result of [12] prompts the question of whether or not the declining price anomaly is guaranteed to hold in general, that is, in sequential auctions with more than two buyers. We answer this question in the negative by exhibiting a sequential auction with three buyers and eight items where prices initially rise and then fall. In order to run our experiments that find this counter-example (to the conjecture that prices are weakly decreasing for multi-buyer sequential auctions) we study in detail the form of equilibria in sequential auctions. First, it is important to note that there is a fundamental distinction between sequential auctions with two buyers and sequential auctions with three or more buyers. In the former sequential auction, each subgame reduces to a standard *auction with independent valuations*. We explain this in Section 2.1, where we present the two-buyer full-information model of [12]. In contrast, in a multi-buyer sequential auction each subgame reduces to an *auction with interdependent valuations*. This is explained in Section 2.2 after we present the extension of the model of [12] to multi-buyer sequential auctions. Consequently to study multi-buyer sequential auctions we must study the equilibria of auctions with interdependent valuations. A theory of such equilibria was recently developed by [24] via a correspondence with an ascending price mechanism. In particular, as we discuss in Section 2.3, this ascending price mechanism outputs a unique bid value, called the *dropout bid* β_i, for each buyer i. For first-price auctions it is known [24] that these dropout bids form a subgame perfect equilibrium and, moreover, the interval $[0, \beta_i]$ is the exact set of bids that survives *all* processes consisting of the iterative deletion of strategies that are weakly dominated. In contrast, we show in Section 2.3 that for second-price auctions it may be the case that no bids survive the iterative deletion of weakly dominated strategies; however, we prove that the interval $[0, \beta_i]$ is the exact set of bids for any losing buyer that survives *all* processes consisting of the iterative deletion of strategies that are weakly dominated *by a lower bid*.

In Section 3 we describe the counter-example. This counter-example, and all of our theorems in Sections 3 and 4, apply to both the first-price and second-price sequential auction settings. We emphasize that there is nothing unusual about our example. The form of the valuation functions used for the buyers is standard, namely, weakly decreasing marginal valuations. Furthermore, the non-monotonic price trajectory does not arise because of the use of an artificial tie-breaking rule; the three most natural tie-breaking rules, see Section 2.4, all induce the same non-monotonic price trajectory. Indeed, we present an even stronger result in Section 4: for *any* tie-breaking rule, there is a sequential auction on which it induces a non-monotonic price trajectory.

This lack of weakly decreasing prices provides an explanation for why multi-buyer sequential auctions have been hard to analyze quantitatively. We provide a second explanation in Section 4.3. There we present a three-buyer sequential auction that does satisfy weakly decreasing prices but which has subgames where some agent has a negative value from winning against one of the two other agents. Again, this contrasts with the two-buyer case where every agent always has a non-negative value from winning against the other agent in every subgame.

Finally in Section 5, we describe the results obtained via our large scale experimentations. These results show that whilst the declining price anomaly is not universal, exceptions are extremely rare. Specifically, from a randomly generated dataset of over six million sequential auctions and a variety of tie-breaking rules only a 0.000183 proportion of the instances produced non-monotonic price trajectories. Consequently, these experiments are consistent with the practical examples discussed in the introduction. Of course, it is perhaps unreasonable to assume that subgame perfect equilibria arise in practice; we remark, though, that the use of simple bidding algorithms by bidders may also lead to weakly decreasing prices in a multi-buyer sequential auction. For example, [26] presents a method called the *residual monopsonist procedure* inducing this property in restricted settings.

2 The Sequential Auction Model

Here we present the full-information sequential auction model. There are T identical items and n buyers. Exactly one item is sold in each time period over T time periods. Buyer i has a value $V_i(k)$ for winning exactly k items. Thus $V_i(k) = \sum_{\ell=1}^{k} v_i(\ell)$, where $v_i(\ell)$ is the marginal value buyer i has for obtaining an ℓth item. This induces an extensive form game. [12], for the two-buyer case, and [24], for the multiple-buyer *first-price* case, show that this sequential game has a focal subgame perfect equilibrium. In this section, we will show that an exact analogue of their results *do not* hold for the *second-price* case with multiple buyers. However, the main purpose of this section is to show that, in effect, the equilibrium of [24] is also a focal equilibrium in the second-price setting, and their results can be extended to this setting with a small technical modification.

To analyze this game it is informative to begin by considering the 2-buyer case, as studied by [12], which we do in the following subsection. [24] show that in the first-price setting, this extensive form game has a focal subga

2.1 The Two-Buyer Case

During the auction, the relevant history is the number of items each buyer has currently won. Thus we may compactly represent the extensive form ("tree") of the auction using a directed graph with a node (x_1, x_2) for any pair of non-negative integers that satisfies $x_1 + x_2 \leq T$. The node (x_1, x_2) induces a subgame with $T - x_1 - x_2$

items for sale and where each buyer i already possesses x_i items. Note there is a *source node*, $(0, 0)$, corresponding to the whole game, and *sink nodes* (x_1, x_2), where $x_1 + x_2 = T$. The values Buyer 1 and Buyer 2 have for a sink node (x_1, x_2) are $\Pi_1(x_1, x_2) = V_1(x_1)$ and $\Pi_2(x_1, x_2) = V_2(x_2)$, respectively. We want to evaluate the values (utilities) at the source node $(0, 0)$. We can do this recursively working from the sinks upwards. Take a node (x_1, x_2), where $x_1 + x_2 = T - 1$. This node corresponds to the final round of the auction, where the last item is sold, given that each buyer i has already won x_i items. The node (x_1, x_2) will have directed arcs to the sink nodes $(x_1 + 1, x_2)$ and $(x_1, x_2 + 1)$; these arcs correspond to Buyer 1 and Buyer 2 winning the final item, respectively. For the case of second-price auctions, it is then a weakly dominant strategy for Buyer 1 to bid its marginal value $v_1(x_1 + 1) = V_1(x_1 + 1) - V_1(x_1)$; similarly for Buyer 2. Of course, this marginal value is just $v_1(x_1 + 1) = \Pi_1(x_1 + 1, x_2) - \Pi_1(x_1, x_2 + 1)$, the difference in value between winning and losing the final item. If Buyer 1 is the highest bidder at (x_1, x_2), that is, $\Pi_1(x_1 + 1, x_2) - \Pi_1(x_1, x_2 + 1) \geq \Pi_2(x_1, x_2 + 1) - \Pi_2(x_1 + 1, x_2)$, then we have that

$$\begin{aligned} \Pi_1(x_1, x_2) &= \Pi_1(x_1 + 1, x_2) - \big(\Pi_2(x_1, x_2 + 1) - \Pi_2(x_1 + 1, x_2)\big) \\ \Pi_2(x_1, x_2) &= \Pi_2(x_1 + 1, x_2) \end{aligned}$$

That is, Buyer 1's value for the node (x_1, x_2), which is $\Pi_1(x_1, x_2)$, is its value for the node $(x_1 + 1, x_2)$ minus Buyer 2's bid for the next item. Symmetric formulas apply if Buyer 2 is the highest bidder at (x_1, x_2). Hence we may recursively define a value for each buyer for each node. The iterative elimination of weakly dominated strategies then leads to a subgame perfect equilibrium [5, 12].

Example Consider a two-buyer sequential auction with two items, where the marginal valuations are $\{v_1(1), v_1(2)\} = \{10, 8\}$ and $\{v_2(1), v_2(2)\} = \{6, 3\}$. This game is illustrated in Fig. 1. The base case with the values of the sink nodes is shown in Fig. 1a. The first row in each node refers to Buyer 1 and shows the number of items won (in plain text) and the corresponding value (in bold); the second row refers to Buyer 2. The outcome of the second-price sequential auction, solved recursively, is then shown in Fig. 1b. Arcs are labelled by the bid value; here arcs for Buyer 1 point left and arcs for Buyer 2 point right. Solid arcs represent winning bids and dotted arcs are losing bids. The equilibrium path is shown in bold.

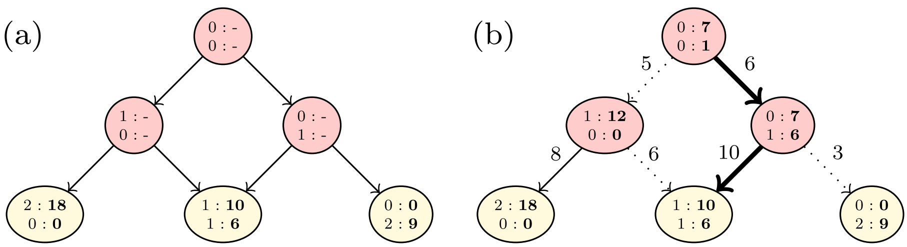


Fig. 1 Second-Price Sequential Auction

Observe that the *declining price anomaly* is exhibited in this example. Specifically, in this subgame perfect equilibrium, Buyer 2 wins the first item for a price 5 and Buyer 1 wins the second item for a price 3. As stated, this example is not an exception. [12] showed that weakly decreasing prices are a property of 2-buyer sequential auctions.

Theorem 2.1 [12] *In a 2-buyer second-price sequential auction there is a unique equilibrium that survives the iterative deletion of weakly dominated strategies. Moreover, at this equilibrium prices are weakly declining.*

We remark that the subgame perfect equilibrium that survives the iterative elimination of weakly dominated strategies is unique in terms of the values at the nodes. Moreover, given a fixed tie-breaking rule, the subgame perfect equilibrium also has a unique equilibrium path in each subgame.

In addition, Theorem 2.1 also applies to first-price sequential auctions. In this case, to ensure the existence of an equilibrium, we make the standard assumption that there is a fixed small bidding increment. That is, for any price p there is a unique maximum price smaller than p. Given this, for the example above, the subgame perfect equilibrium using a first-price sequential auction is as shown in Fig. 2. Here we use the notation p^+ to denote a winning bid of value equal to p, and the notation p to denote a losing bid equal to maximum value smaller than p.

Observe that the resultant prices on the equilibrium path are more easily apparent in Fig. 2 than in Fig. 1. For this reason, all the figures we present in the rest of the paper will be for first-price auctions; equivalent figures can can be drawn for the case of second-price auctions.

So the decreasing price anomaly holds in two-buyer sequential auctions. The question of whether or not it applies to sequential auctions with more than two buyers remained open prior to this work. We resolve this question in the rest of this paper. To do this, let's first study equilibria in the full-information sequential auction model when there are more than two buyers.

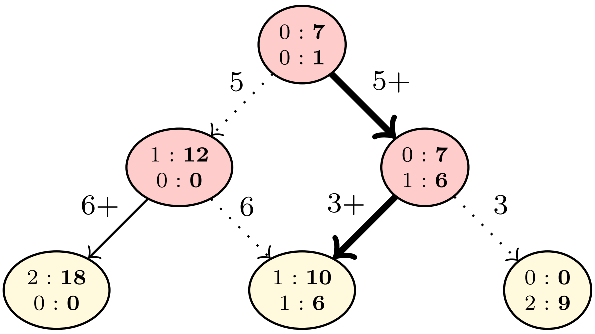


Fig. 2 First-Price Sequential Auction

2.2 The Multi-Buyer Case

The underlying model of [12] extends simply to sequential auctions with $n \geq 3$ buyers. There is a node $(x_1, x_2, \ldots, x_n)$ for each set of non-negative integers satisfying $\sum_{i=1}^{n} x_i \leq T$. There is a directed arc from $(x_1, x_2, \ldots, x_n)$ to $(x_1, x_2, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots x_n)$ for each $1 \leq j \leq n$. Thus each non-sink node has n out-going arcs. This is problematic: whilst in the final time period each buyer has a value for winning and a value for losing, this is no longer the case recursively in earlier time periods. Specifically, buyer i has value for winning, but $n-1$ (different) values for losing depending upon the identity of the buyer $j \neq i$ who actually wins. Thus rather than each node corresponding to a standard auction, each node in the multi-buyer case corresponds to an auction with interdependent valuations.

Formally, an *auction with interdependent valuations* is a single-item auction where each buyer i has a value $v_{i,i}$ for winning the item and, for each buyer $j \neq i$, buyer i has value $v_{i,j}$ if buyer j wins the item. These auctions, also called *auctions with externalities*, were introduced by [11] and by [17]. Their motivations were applications where losing participants were not indifferent to the identity of the winning buyer; examples include firms seeking to purchase a patented innovation, take-over acquisitions of a smaller company in an oligopolistic market, and sports teams competing to sign a star athlete.

Therefore to understand multi-buyer sequential auctions we must first understand equilibria in auctions with interdependent valuations. This is actually not a simple task. Indeed such an understanding was only recently provided by [24].

2.3 Equilibria in Auctions with Interdependent Valuations

2.3.1 An Ascending Price Mechanism

We can explain the result of [24] via an ascending price auction. Imagine a two-buyer ascending price auction where the valuations of the buyers are $v_1 > v_2$. The requested price p starts at zero and continues to rise until the point where the second buyer drops out. Of course, this happens when the price reaches v_2, and so Buyer 1 wins for a payment $p^+ = v_2$. But this is exactly the outcome expected from a first-price auction: Buyer 2 loses with bid of p and Buyer 1 wins with a bid of p^+. To generalize this to multi-buyer settings we can view this process as follows. At a price p, buyer i remains in the auction as long as there is at least one buyer j *still in the auction* who buyer i is willing to pay a price p to beat; that is, $v_{i,i} - p > v_{i,j}$. The last buyer to drop out wins at the corresponding price. For example, in the two-buyer example above, Buyer 2 drops out at price $p = v_2$ as it would rather lose to Buyer 1 than win above that price. Therefore, at price p^+ there is no buyer still in the auction that Buyer 1 wishes to beat (because there are no other buyers remaining in the auction at all!). Thus Buyer 1 drops out at p^+ and, being the last buyer to drop out, wins at that price.

Observe that, even in the multi-buyer setting, this procedure produces a unique *dropout bid* β_i for each buyer i. To illustrate this, two auctions with interdependent valuations are shown in Fig. 3. In these diagrams the label of an arc from buyer i to

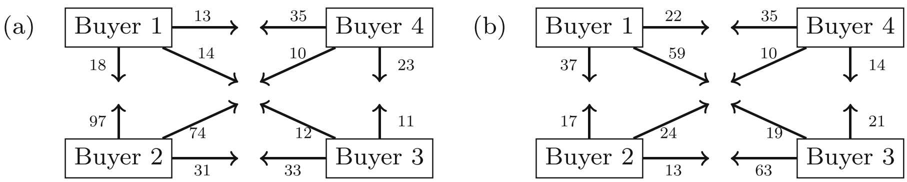


Fig. 3 DROP-OUT BID EXAMPLES. *In these two examples the dropout bid vectors* $(\beta_1, \beta_2, \beta_3, \beta_4)$ are $(18, 31, 31^+, 23)$ and $(24, 24, 24, 24^+)$, respectively

buyer j is $w_{i,j} = v_{i,i} - v_{i,j}$. That is, buyer i is willing to pay up to $w_{i,j}$ to win *if the alternative is that buyer j wins the item*. Now consider running our ascending price procedure for these auctions. In Fig. 3a, Buyer 1 drops out when the price reaches 18. Since Buyer 1 is no longer active in the auction, Buyer 4 drops out when the price reaches 23. At this point, Buyer 2 and Buyer 3 are left to compete for the item. Buyer 3 wins when Buyer 2 drops out at price 31. Thus the drop-out bid of Buyer 3 is 31^+. Observe that Buyer 2 loses despite having very high values for winning (against Buyer 1 and Buyer 4).

The example of Fig. 3b with dropout bid vector $(\beta_1, \beta_2, \beta_3, \beta_4) = (24, 24, 24, 24^+)$ is more subtle. Here Buyer 2 drops out at price 24. But Buyer 3 only wanted to beat Buyer 2 at this price so it then immediately drops out at the same price. Now Buyer 1 only wanted to beat Buyer 2 and Buyer 3 at this price, so it then immediately drops out at the same price. This leaves Buyer 4 the winner at price 24^+.

2.3.2 Dropout Bids and Iterative Deletion of Weakly Dominated Strategies

As well as being solutions to the ascending price auction, the dropout bids have a much stronger property that makes them the natural and robust prediction for auctions with interdependent valuations. Specifically, [24] proved that, for each buyer i, the interval $[0, \beta_i]$ is the set of strategies that survive *any sequence* consisting of the iterative deletion of weakly dominated strategies. This is formalized as follows. Take an n-buyer game with strategy sets $S_1, S_2, \ldots, S_n$ and utility functions $u_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$. Then $\{S_i^\tau\}_{i,\tau}$ is a *valid sequence* for the iterative deletion of weakly dominated strategies if for each τ there is a buyer i such that $(i) S_j^\tau = S_j^{\tau-1}$ for each buyer $j \neq i$ and $(ii) S_i^\tau \subset S_i^{\tau-1}$ where for each strategy $s_i \in S_i^{\tau-1} \setminus S_i^\tau$ there is an $\hat{s}_i \in S_i^\tau$ such that $u_i(\hat{s}_i, s_{-i}) \geq u_i(s_i, s_{-i})$ for all $s_{-i} \in \prod_{j:j\neq i} S_j^\tau$, and with strict inequality for at least one s_{-i}.

We say that a strategy s_i for buyer i *survives* the iterative deletion of weakly dominated strategies if for any valid sequence $\{S_i^\tau\}_{i,\tau}$ we have $s_i \in \bigcap_\tau S_i^\tau$.

Theorem 2.2 [24] *Given a first-price auction with interdependent valuations, for each buyer i, the set of bids that survive the iterative deletion of weakly dominated strategies is exactly* $[0, \beta_i]$.

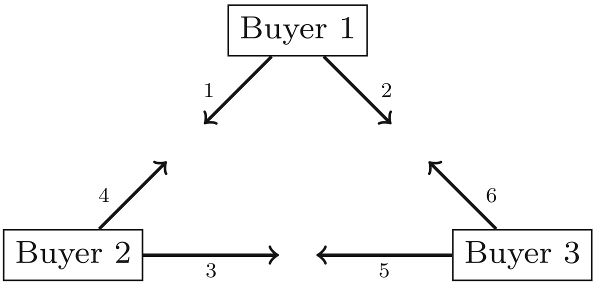


Fig. 4 *A second-price auction with interdependent valuations where no strategies survive the iterative deletion of weakly dominated strategies*

An exact analogue of Theorem 2.2 does *not* hold for second-price auctions with interdependent valuations. Instead, it may be the case that no strategies survive the iterative deletion of weakly dominated strategies.

Theorem 2.3 *There are second-price auctions with interdependent valuations where no strategies survive the iterative deletion of weakly dominated strategies.*

Proof Consider the 3-buyer auction with interdependent valuations shown in Fig. 4.

Let's now examine what happens when we use the two different orderings illustrated in Fig. 5 to delete weakly dominated strategies.

Consider first the iterative process in the first table of Fig. 5. Observe that Buyer 3 is willing to pay 6 to beat Buyer 1 and 5 to beat Buyer 2. It follows that any bid above 6 is weakly dominated by a bid of 6. Moreover, as this is a second-price auction, any bid below 5 is weakly dominated by a bid of 5 (we emphasize that this latter fact does not hold in the case of first-price auctions). Now, rather than deleting all these bids immediately, imagine that Buyer 3 deletes any bid over 6 *and* any bid between 1 and 2. Therefore $S_3^1 = [0, 6] \setminus [1, 2]$. At this stage the undeleted strategies for Buyer 1 and Buyer 2 remain $S_1^1 = S_2^1 = [0, \infty)$.

	Buyer 1	Buyer 2	Buyer 3
S^0	$[0, \infty)$	$[0, \infty)$	$[0, \infty)$
S^1	$[0, \infty)$	$[0, \infty)$	$[0, 6] \setminus [1, 2]$
S^2	$\{1\}$	$[0, \infty)$	$[0, 6] \setminus [1, 2]$

	Buyer 1	Buyer 2	Buyer 3
S^0	$[0, \infty)$	$[0, \infty)$	$[0, \infty)$
S^1	$[0, \infty)$	$[0, 4] \setminus [1, 2]$	$[0, \infty)$
S^2	$\{2\}$	$[0, 4] \setminus [1, 2]$	$[0, \infty)$

Fig. 5 *Two processes that together eliminate every strategy for Buyer 1*

	Buyer 1	Buyer 2	Buyer 3
S^0	$[0, \infty)$	$[0, \infty)$	$[0, \infty)$
S^1	$[0, \infty)$	$[0, \infty)$	$[0, 6] \setminus [3, 4]$
S^2	$[0, \infty)$	$\{4\}$	$[0, 6] \setminus [3, 4]$

	Buyer 1	Buyer 2	Buyer 3
S^0	$[0, \infty)$	$[0, \infty)$	$[0, \infty)$
S^1	$[0, 2]$	$[0, \infty)$	$[0, \infty)$
S^2	$[0, 2]$	$\{3\}$	$[0, \infty)$

Fig. 6 *Two processes that together eliminate every strategy for Buyer 2*

Next observe that Buyer 1 is willing to pay at most 1 to beat Buyer 2 but up to 2 to beat Buyer 3. Because this is a second-price auction, it immediately follows that any bid below one or above two is weakly dominated. Now let's compare the outcomes for Buyer 1 between bidding 1 and bidding $x \in (1, 2]$. If Buyer 2 and Buyer 3 are both bidding below one then Buyer 1 wins with a bid of 1 or a bid of x. If either Buyer 2 or Buyer 3 is bidding greater than x then Buyer 1 loses with a bid of 1 or a bid of x. So suppose the highest bid from Buyer 2 and Buyer 3 is between 1 and x. If this highest bid is from Buyer 2 then Buyer 1 would prefer to lose and so bidding 1 is preferable to bidding x. On the other hand, if this highest bid is from Buyer 3 then Buyer 1 would prefer to win and so bidding x is preferable to bidding 1. But the latter case cannot happen as the strategy space of Buyer 3 is currently $S_3^1 = [0, 6] \setminus [1, 2]$. It follows that bidding 1 weakly dominates bidding x. Hence we can set $S_1^2 = \{1\}$.

Next consider the iterative process in the second part of Fig. 5. This time let's begin by deleting strategies of Buyer 2 that are weakly dominated. Observe that Buyer 2 is willing to pay 4 to beat Buyer 1 and 3 to beat Buyer 3. Let's imagine that Buyer 2 now deletes any bid over 4 *and* any bid between 1 and 2. Therefore $S_2^1 = [0, 4] \setminus [1, 2]$. At this stage the undeleted strategies for Buyer 1 and Buyer 3 remain $S_1^1 = S_3^1 = [0, \infty)$.

In the next step consider Buyer 1. Again, bidding less than one or above two is weakly dominated. This time let's compare the outcomes for Buyer 1 between bidding 2 and bidding $x \in [1, 2)$. If Buyer 2 and Buyer 3 are both bidding below x then Buyer 1 wins with a bid of 2 or a bid of x. If either Buyer 2 or Buyer 3 is bidding greater than 2 then Buyer 1 loses with a bid of 2 or a bid of x. So suppose the highest bid from Buyer 2 and Buyer 3 is between x and 2. If this highest bid is from Buyer 2 then Buyer 1 would prefer to lose and so bidding x is preferable to bidding 2. On the other hand, if this highest bid is from Buyer 3 then Buyer 1 would prefer to win and so bidding 2 is preferable to bidding x. But the former case cannot happen as the strategy space of Buyer 2 is currently $S_2^1 = [0, 4] \setminus [1, 2]$. If follows that bidding 2 weakly dominates bidding x. Hence we can set $S_1^2 = \{2\}$.

But $\{1\} \cap \{2\} = \emptyset$. Therefore, *no* strategy for Buyer 1 survives the iterative deletion of weakly-dominated strategies. That is, for each bid value b there is a sequence of iterative deletions of weakly-dominated strategies that deletes the bid b of Buyer 1.

We remark that, for this example, similar arguments show that no bid for Buyer 2 survives the iterative deletion of weakly-dominated strategies. In particular, the two processes shown in Fig. 6 lead to non-intersecting, undeleted, strategy sets for Buyer 2.

Consideration of the above example shows that the problem occurs when a strategy is deleted because it is weakly dominated by a *higher* value bid. Observe that this can never happen for a potentially winning bid in a first-price auction. Thus Theorem 2.2 still holds in first-price auctions when we restrict attention to sequences consisting of the iterative deletion of strategies that are weakly dominated by a *lower* bid. Indeed, we can prove the corresponding theorem also holds for second-price auctions.

Theorem 2.4 *Given a second-price auction with interdependent valuations, for each losing buyer i, the set of bids that survive the iterative deletion of strategies that are weakly dominated by a lower bid is exactly* $[0, \beta_i]$.

Proof First we claim that for any losing buyer i and any price $p > \beta_i$ there is a sequence of iterative deletions of strategies that are weakly dominated by a lower bid that leads to the deletion of bid p from S_i^τ. Without loss of generality, we may order the buyers such that $\beta_1 \le \beta_2 \le \cdots \le \beta_n$; in the case of a tie the buyers are placed in the order they were deleted by the tie-breaking rule. Initially $S_i^0 = [0, \infty)$, for each buyer i. We now define a valid sequence such that $S_i^i = [0, \beta_i]$. We proceed by induction on the label of the buyers. For the base case observe that for Buyer 1 we know $\beta_1 = \max_{j:j\neq i}(v_{i,i} - v_{i,j})$ is the highest price it wants to pay to beat anyone else. Suppose Buyer 1 bids $p > \beta_1$. Take any set of bids $b_{-1} \in \times_{j:j\ge 2} S_j^0$. We have three cases:

- (i) Both bids p and β_1 are winning bids against b_{-1}. Then, as this is a second-price auction, Buyer 1 is indifferent between the two bids.
- (ii) Both bids p and β_1 are losing bids against b_{-1}. Then Buyer 1 is indifferent between the two bids.
- (iii) Bid p is a winning bid but β_1 is a losing bid against b_{-1}. Then since the winning price is at least β_1, Buyer 1 strictly prefers to lose rather than win. Moreover, since $S_j^0 = [0, \infty)$, there is a set of bids b_{-1} by the other buyers such that Buyer 1 strictly prefers to lose rather than win.

Thus the bid p is weakly dominated by the lower bid β_1. Since this applies to any $p > \beta_1$, in Step 1 we may delete every bid for Buyer 1 above β_1. Therefore $S_1^1 = [0, \beta_1]$ and $S_j^1 = [0, \infty]$ for each buyer $j \ge 2$.

For the induction hypothesis assume $S_j^{i-1} = [0, \beta_j]$, for all $j < i$ and $S_j^{i-1} = [0, \infty)$, for all $j \ge i$. Now take a losing buyer i and any set of bids $b_{-i} \in \times_{j:j\neq i} S_j^{i-1}$. Again, we have three cases:

- (i) Both bids p and β_i are winning bids against b_{-i}. Then, as this is a second-price auction, buyer i is indifferent between the two bids.

(ii) Both bids p and β_1 are losing bids against b_{-i}. Then buyer i is indifferent between the two bids.

(iii) Bid p is a winning bid but β_i is a losing bid against b_{-i}. Then since β_i is a losing bid under the tie-breaking rule, it must be the case that the winning bid is from a buyer j where $j > i$. But, by definition of β_i, there is no buyer j, with $j > i$, that buyer i wishes to beat at price β_i.

So buyer i prefers the bid β_i to the bid p. Moreover, since any buyer $j : j > i$ has $S_j^{i-1} = [0, \infty)$, this preference is strict for some feasible choice of bids for the other buyers. Thus, for buyer i, the bid p is weakly dominated by the lower bid β_i, and this applies to every $p > \beta_i$. Ergo, in Step i we may delete every bid for buyer i above β_i. Therefore $S_j^i = [0, \beta_i]$, for all $j < i + 1$ and $S_j^{i-1} = [0, \infty)$, for all $j \geq i + 1$. The claim then follows by induction. So, for any losing buyer i we have that no bid greater than β_i survives the iterative deletion of strategies that are weakly dominated by a lower bid.

Observe that the above arguments also apply for the winning buyer, that is, buyer n. Except, as there are no higher indexed buyers, it is not the case that β_n strictly dominates any bid $p > \beta_n$. Indeed, buyer n is indifferent between all bids in the range $[\beta_n, \gamma_n]$, where γ_n is the maximum value the buyer has for beating any buyer j with dropout bid $\beta_j = \beta_n$. Observe, γ_n does exist and is at least β_n by definition of the ascending price mechanism. Thus, for the winning bidder no bid greater than γ_i survives the iterative deletion of strategies that are weakly dominated by a lower bid.

Second, we claim for any buyer i and any price $q < \beta_i$ there is no sequence of iterative deletions of strategies that are weakly dominated by a lower bid that leads to the deletion of bid q from the feasible strategy space of buyer i. If not, consider the first time τ that some buyer i has a value $q \in [0, \beta_i]$ deleted from S_i^τ. We may assume that q is deleted because it is was weakly dominated by a lower bid $p < q$. Now, by assumption, $[0, \beta_j] \subseteq S_j^{\tau-1}$, for each buyer j. Furthermore, by definition, there is some buyer k, with $k > i$ that buyer i wishes to beat at any price below β_i. In particular, Buyer i wishes to beat Buyer k at price p. But since $k > i$ we have $\beta_k \geq \beta_i$. Recall that $[0, \beta_k] \subseteq S_k^{\tau-1}$. It immediately follows that there is a set of feasible bids $b_k \in (p, q)$ and $b_j = 0$, for all $j \notin \{i, k\}$ such that Buyer i strictly prefers to win against these bids. Specifically, the bid q is not weakly dominated by the bid p, a contradiction.

It follows that the dropout bids form the *focal* subgame perfect equilibrium for both first-price and second-price auctions with interdependent valuations.

We are now almost ready to be able to find equilibria in the sequential auction experiments we will conduct. This, in turn, will allow us to present a sequential auction with non-monotonic prices. Before doing so, one final factor remains to be discussed regarding the transition from equilibria in auctions with interdependent valuations to equilibria in sequential auctions.

2.4 Equilibria in Sequential Auctions

2.4.1 Tie-Breaking Rules

As stated, the dropout bid of each buyer is uniquely defined. However, our description of the ascending auction may leave some flexibility in the choice of winner. Specifically, it may be the case that simultaneously more than one buyer wishes to drop out of the auction. If this happens at the end of the ascending price procedure then any of these buyers could be selected as the winner. An example of this is shown in Fig. 7.

This observation implies that to fully define the ascending auction procedure we must incorporate a tie-breaking rule to order the buyers when more than one wish to drop out simultaneously. In a one-shot auction with interdependent valuations the tie-breaking rule only affects the choice of winner, but otherwise has no structural significance. However, in a sequential auction the choice of tie-breaking rule may have much more significant consequences. Specifically, because each node in the game tree corresponds to an auction with interdependent valuations, the choice of winner at one node may affect the valuations at nodes higher in the tree. In particular, the equilibrium path may vary with different tie-breaking rules, leading to different prices, winners, and utilities.

As we will show in Section 4.1 there are a massive number of tie-breaking rules, even in small sequential auctions. We emphasize, however, that our main result holds regardless of the tie-breaking rule. That is, for *any* tie-breaking rule there is a sequential auction on which it induces a non-monotonic price trajectory. This we will also show in Section 4 after explaining mathematically how to classify every tie-breaking rule in terms of labelled, directed acyclic graphs. First, though, we will show that non-monotonic pricing occurs on the equilibrium path for perhaps the three most natural choices of tie-breaking rule, namely `preferential-ordering`, `first-in-first-out` and `last-in-first-out`. Interestingly these rules correspond to the fundamental data structures of priority queues, queues, and stacks used in computer science.

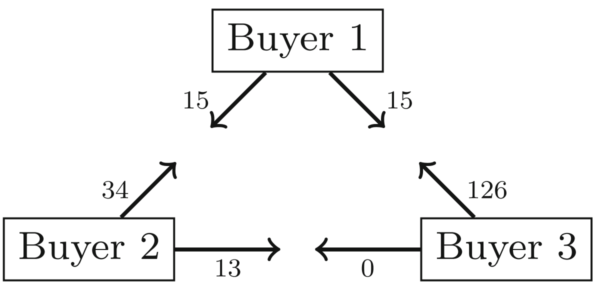


Fig. 7 TIE-BREAKING. *An example requiring tie-breaking to decide the winner. The drop-out bid vector is* (15, 15, 15) *but there are two possible winners:* $(15, 15^+, 15)$ or $(15, 15, 15^+)$

2.4.2 Tie-Breaking Rules (and Data Structures!)

Preferential Ordering (Priority Queue) In `preferential-ordering` each buyer is given a distinct rank. In the case of a tie the buyer with the worst rank is eliminated. Without loss of generality, we may assume that the ranks corresponding to a lexicographic ordering of the buyers. That is, the rank of a buyer is its index label and given a tie amongst all the buyers that wish to dropout of the auction we remove the buyer with the highest index. The preferential ordering tie-breaking rule corresponds to the data structure known as a *priority queue*.

First-In-First-Out (Queue) The `first-in-first-out` tie-breaking rule corresponds to the data structure known as a *queue*. The queue consists of those buyers in the auction that wish to dropout. Amongst these, the buyer at the front of the queue is removed. If multiple buyers request to be added to the queue simultaneously, they will be added lexicographically. Note though that this is different from preferential ordering as the entire queue will not, in general, be ordered lexicographically. For example, when at a fixed price p we remove the buyer i at the front of the queue this may cause new buyers to wish to dropout at price p (i.e. those buyers who only wanted to beat buyer i). These new buyers will be placed behind the other buyers already in the queue.

Last-In-First-Out (Stack) The `last-in-first-out` tie-breaking rule corresponds to the data structure known as a *stack*. Again the stack consists of those buyers in the auction that wish to dropout. Amongst these, the buyer at the top of the stack (i.e. the back of the queue) is removed. If multiple buyers request to be added to the stack simultaneously, they will be added lexicographically. At first glance, this `last-in-first-out` rule appears more unusual than the previous two, but it still has a natural interpretation in terms of an auction. Namely, it corresponds to settings where the buyer whose situation has changed most recently reacts the quickest.

In order to understand these tie-breaking rules it is useful to see how they apply on an example. In Fig. 8 the dropout vector is $(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5) = (40, 40, 40, 40, 40)$, but the three tie-breaking rules will select three different winners.

On running the ascending price procedure, both Buyer 3 and Buyer 4 wish to drop out when the price reaches 40. In `preferential-ordering`, the set of agents eligible for dropping out is then $\{3, 4\}$ and we remove the highest index buyer, namely Buyer 4. With the removal of Buyer 4, neither Buyer 1 nor Buyer 5 have an incentive to continue bidding so they both decide to dropout. Thus the set of agents eligible for dropping out is now $\{1, 3, 5\}$ and `preferential-ordering` removes Buyer 5. Observe, with the removal of Buyer 5, that Buyer 2 no longer has an active participant it wishes to beat so the set of agents eligible for dropping out is updated to $\{1, 2, 3\}$. The `preferential-ordering` rule now removes the buyers in the order Buyer 3, then Buyer 2 and lastly Buyer 1. Thus Buyer 1 wins under the `preferential-ordering` rule.

Now consider `first-in-first-out`. To allow for a consistent comparison between the three methods, we assume that when multiple buyers are simultaneously added to the queue they are added in decreasing lexicographical order. Thus our

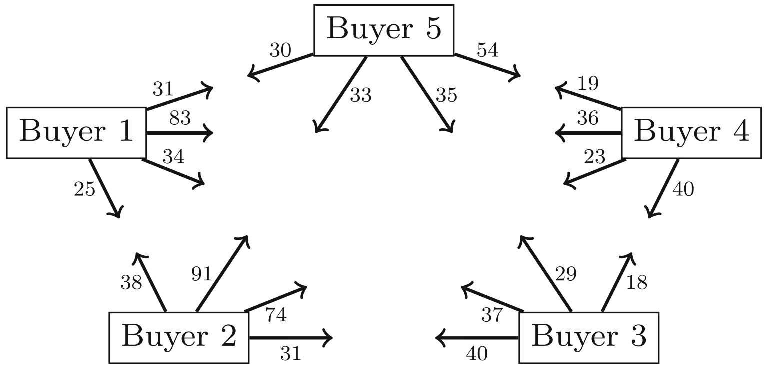


Fig. 8 An Example to Illustrate the Three Tie-Breaking Rules

initial queue is 4 : 3 and `first-in-first-out` removes Buyer 4 from the front of the queue. With the removal of Buyer 4, neither Buyer 1 nor Buyer 5 have an incentive to continue bidding so they are added to the back of the queue. Thus the queue is now 3 : 5 : 1 and `first-in-first-out` removes Buyer 3 from the front of the queue. It then removes Buyer 5 from the front of the queue. With the removal of Buyer 5, we again have that Buyer 2 now wishes to dropout. Hence the queue is 1 : 2 and `first-in-first-out` then removes Buyer 1 from the front of the queue and lastly removes Buyer 2. Thus Buyer 2 wins under the `first-in-first-out` rule.

Finally, consider the `last-in-first-out` rule. Again, to allow for a consistent comparison we assume that when multiple buyers are simultaneously added to the stack they are added in increasing lexicographical order. Thus our initial stack is $\begin{array}{c}4\\3\end{array}$ and `last-in-first-out` removes Buyer 4 from the top of the stack. Again, Buyer 1 and Buyer 5 both now wish to drop out so our stack becomes $\begin{array}{c}5\\1\\3\end{array}$. Therefore Buyer 5 is next removed from the the top of the stack. At this point, Buyer 2 wishes to dropout so the stack becomes $\begin{array}{c}2\\1\\3\end{array}$. The `last-in-first-out` rule now removes the buyers in the order Buyer 2, then Buyer 1 and lastly Buyer 3. Thus Buyer 3 wins under the `last-in-first-out` rule.

We have now developed all the tools required to implement our sequential auction experiments. We describe these experiments and their results in Section 5. Before doing so, we present in Section 3 one sequential auction obtained via these experiments and verify that it leads to a non-monotonic price trajectory with each of the three tie-breaking rules discussed above. We then explain in Section 4 how to generalize this conclusion to apply to every tie-breaking rule.

3 An Auction with Non-Monotonic Prices

Here we prove that the decreasing price anomaly is **not** guaranteed for sequential auctions with more than two buyers. Specifically, in Section 4 we prove the following result:

Theorem 4.1 *For any tie-breaking rule τ, there is a sequential auction on which it produces non-monotonic prices.*

This result is rather surprising, since the result of [12] implies that with two buyers, prices always decrease at equilibrium even when the valuation functions do not have decreasing marginals. In stark contrast, with three or more buyers, the prices can increase along the equilibrium path *despite* the assumption of decreasing marginal valuations.

Before we prove this general result, however, in the rest of this section, we show that for all three of the tie-breaking rules discussed (namely, `preferential-ordering`, `first-in-first-out` and `last-in-first-out`) there is a sequential auction with non-monotonic prices. Specifically, we exhibit a sequential auction with three buyers and eight items that exhibits non-monotonic prices.

Theorem 3.1 *There is a sequential auction with a non-monotonic price trajectory for the `preferential-ordering`, `first-in-first-out` and `last-in-first-out` rules.*

Proof Our counter-example to the conjecture is a sequential auction with three buyers and eight identical items for sale. We present the first-price version where at equilibrium the buyers bid their dropout values in each time period; as discussed, the same example extends to second-price auctions.

The valuations of the three buyers are defined as follows. Buyer 1 has marginal valuations {55, 55, 55, 55, 30, 20, 0, 0}, Buyer 2 has marginal valuations {32, 20, 0, 0, 0, 0, 0, 0}, and Buyer 3 has marginal valuations {44, 44, 44, 44, 0, 0, 0, 0}.

Let's now compute the extensive forms of the auction under the three tie-breaking rules. We begin with the `preferential-ordering` rule. To compute its extensive form, observe that Buyer 1 is guaranteed to win at least two items in the auction because Buyer 2 and Buyer 3 together have positive value for six items. Therefore, the feasible set of sink nodes in the extensive form representation are shown in Fig. 9.

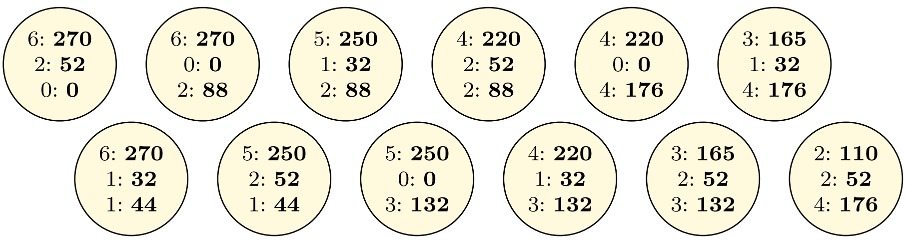


Fig. 9 Sink Nodes of the Extensive Form Game

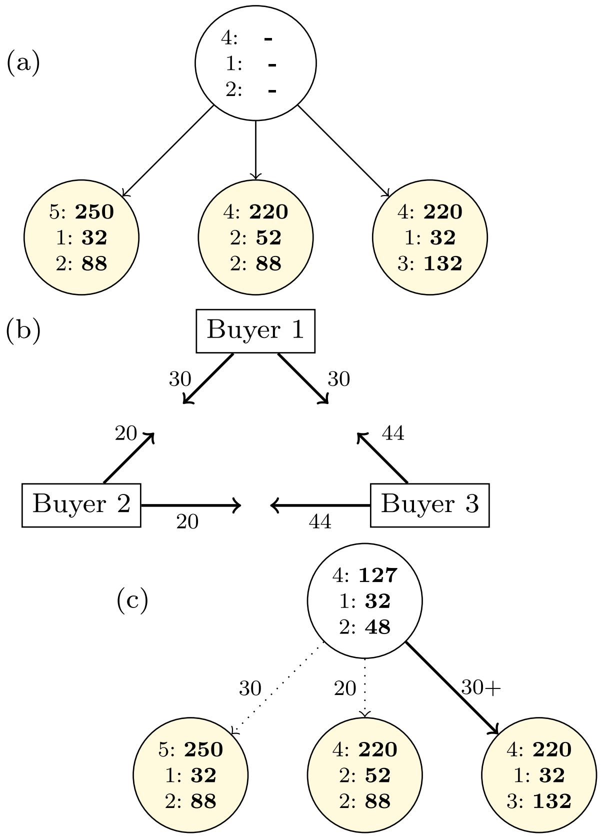


Fig. 10 Solving a Subgame above the Sinks

Given the valuations at the sink nodes we can work our way upwards recursively calculating the values at the other nodes in the extensive form representation. For example, consider the node $(x_1, x_2, x_3) = (4, 1, 2)$. This node has three children, namely (5, 1, 2), (4, 2, 2) and (4, 1, 3); see Fig. 10a. These induce a three-buyer auction as shown in Fig. 10b. This can be solved using the ascending price procedure to find the dropout bids for each buyer. Thus we obtain that the value for the node $(x_1, x_2, x_3) = (4, 1, 2)$ is as shown in Fig. 10c. Of course this node is particularly simple as, for the final round of the sequential auction, the corresponding auction with interdependent valuations is just a standard auction. That is, when the final item is sold, for any buyer i the value $v_{i,j}$ is independent of the buyer $j \neq i$.

Nodes higher up the game tree correspond to more complex auctions with interdependent valuations. For example, the case of the source node $(x_1, x_2, x_3) = (0, 0, 0)$ is shown in Fig. 11. In this case, on applying the ascending price procedure, Buyer 1 is the first to dropout at price 15. At this point, both Buyer 2 and Buyer 3 no longer have a competitor that they wish to beat at this price, so they both want to dropout. With the `preferential-ordering` tie-breaking rule, Buyer 2 wins the item.

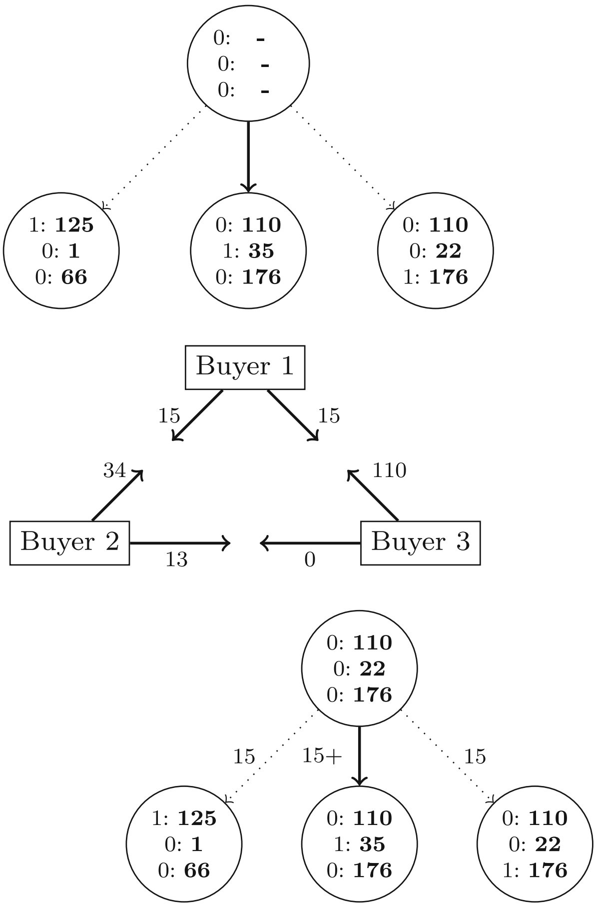


Fig. 11 Solving the Subgame at the Root

Using similar arguments at each node verifies the concise extensive form representation under the `preferential-ordering` tie-breaking rule shown in Fig. 12. In this figure, the white nodes represent subgames where the sequential auction still has three active buyers; the pink nodes represent subgames with at most two active buyers; the yellow nodes are the sink nodes. Again, the equilibrium path with non-monotonic prices is shown in bold. Now consider this equilibrium path. Observe that Buyer 2 wins the first two items, Buyer 3 wins the next four items and Buyer 1 wins the final two items. The resultant price trajectory is {15, 17, 0, 0, 0, 0, 0, 0}. That is, the price rises and then falls to zero – a non-monotonic price trajectory.

Exactly the same example works with the other two tie-breaking rules. The extensive form representation with the `first-in-first-out` rule is shown in Fig. 13; the extensive form representation with the `last-in-first-out` rule is shown in Fig. 14. Notice the node values under `preferential-ordering` and first-in-first-out are exactly the same. This is despite the fact that these two rules do produce different winners at some nodes, for example the node (3, 0, 2). In contrast, the last-in-first-out rule gives an extensive form where some

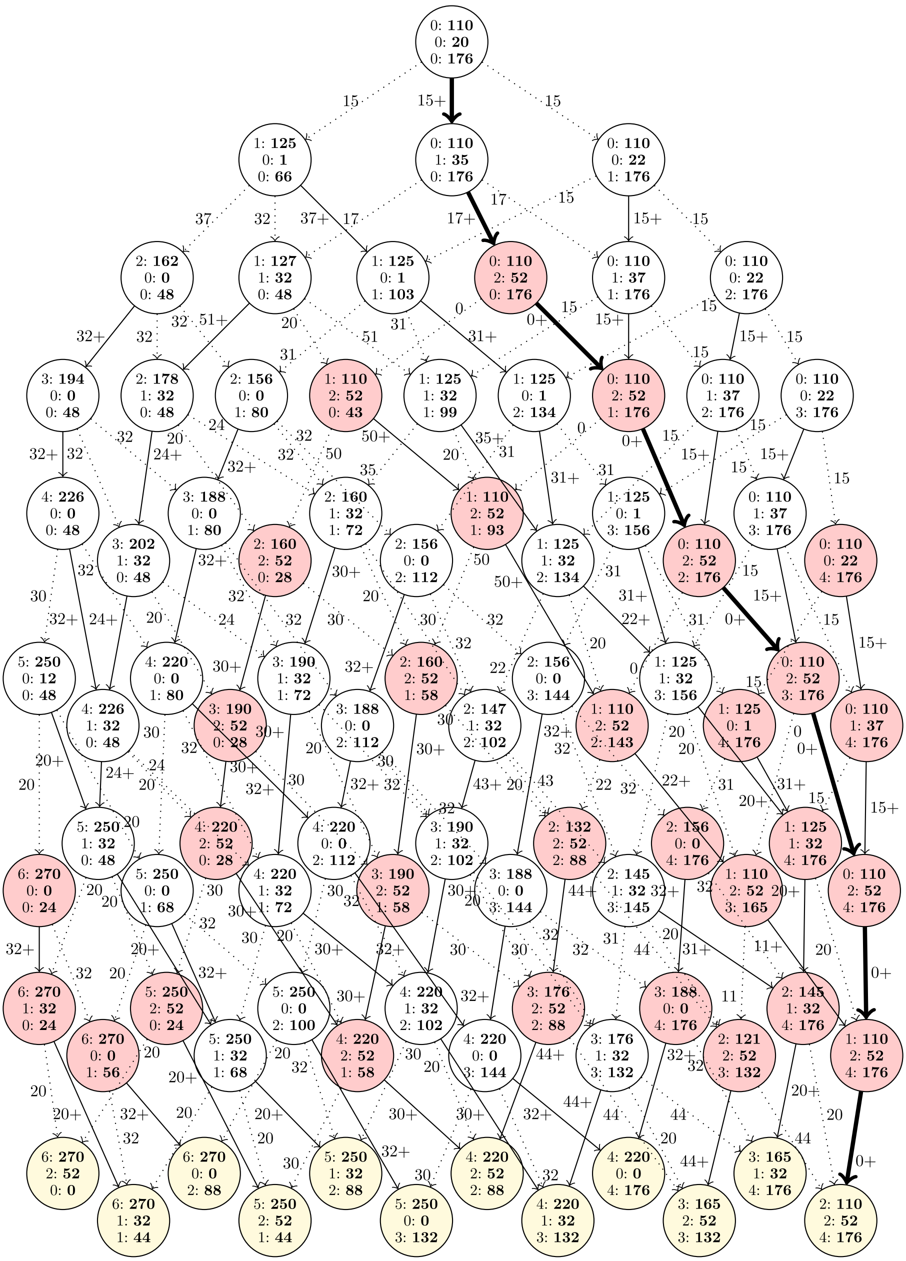


Fig. 12 Non-Monotonic Prices with the preferential-ordering Rule

nodes have different valuations than those produced by the other two rules. For example, for the node (2, 0, 0) and its subgame the equilibrium paths and their prices differ in Figs. 13 and 14. However, for all three rules the equilibrium path and price

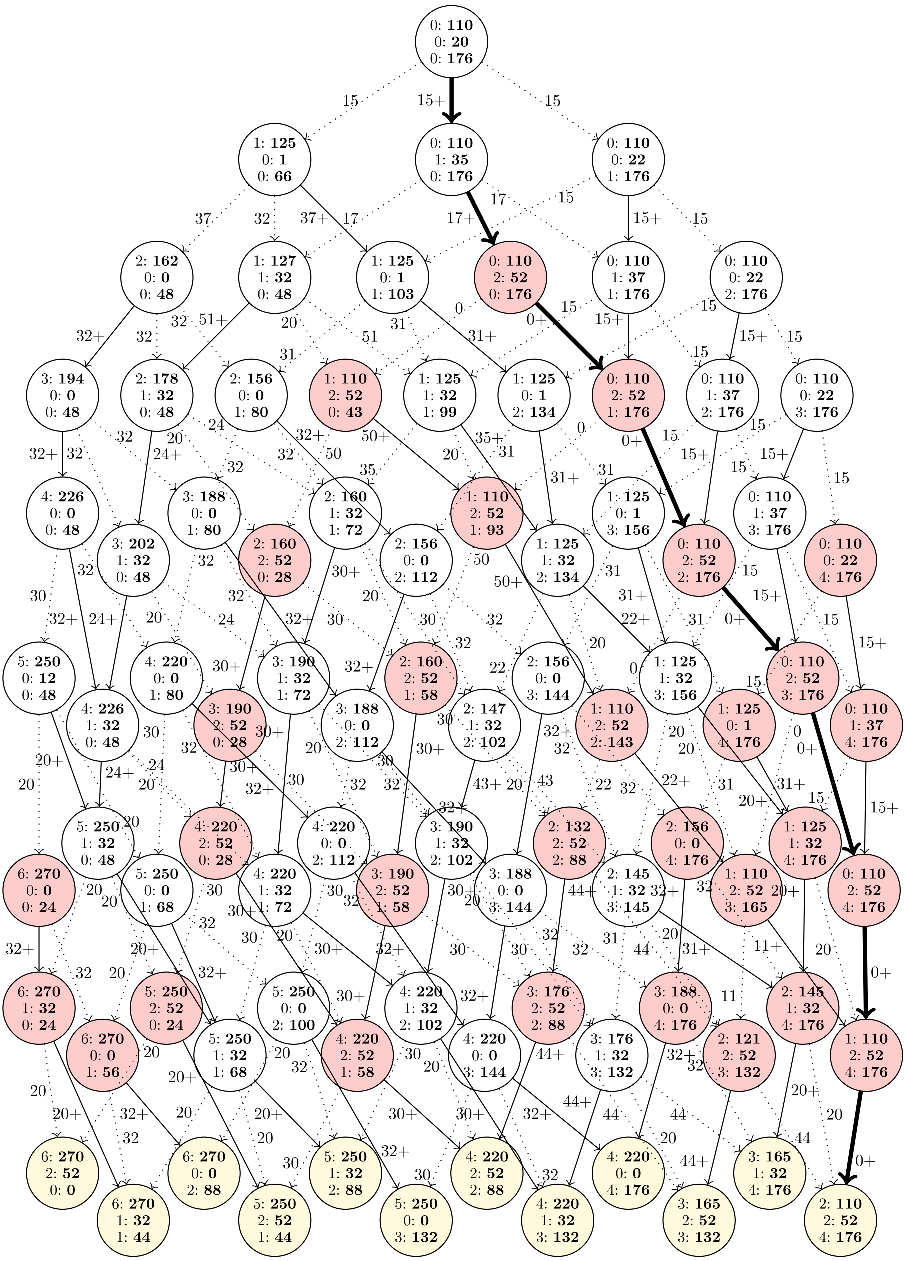


Fig. 13 Non-Monotonic Prices with the `first-in-first-out` Rule

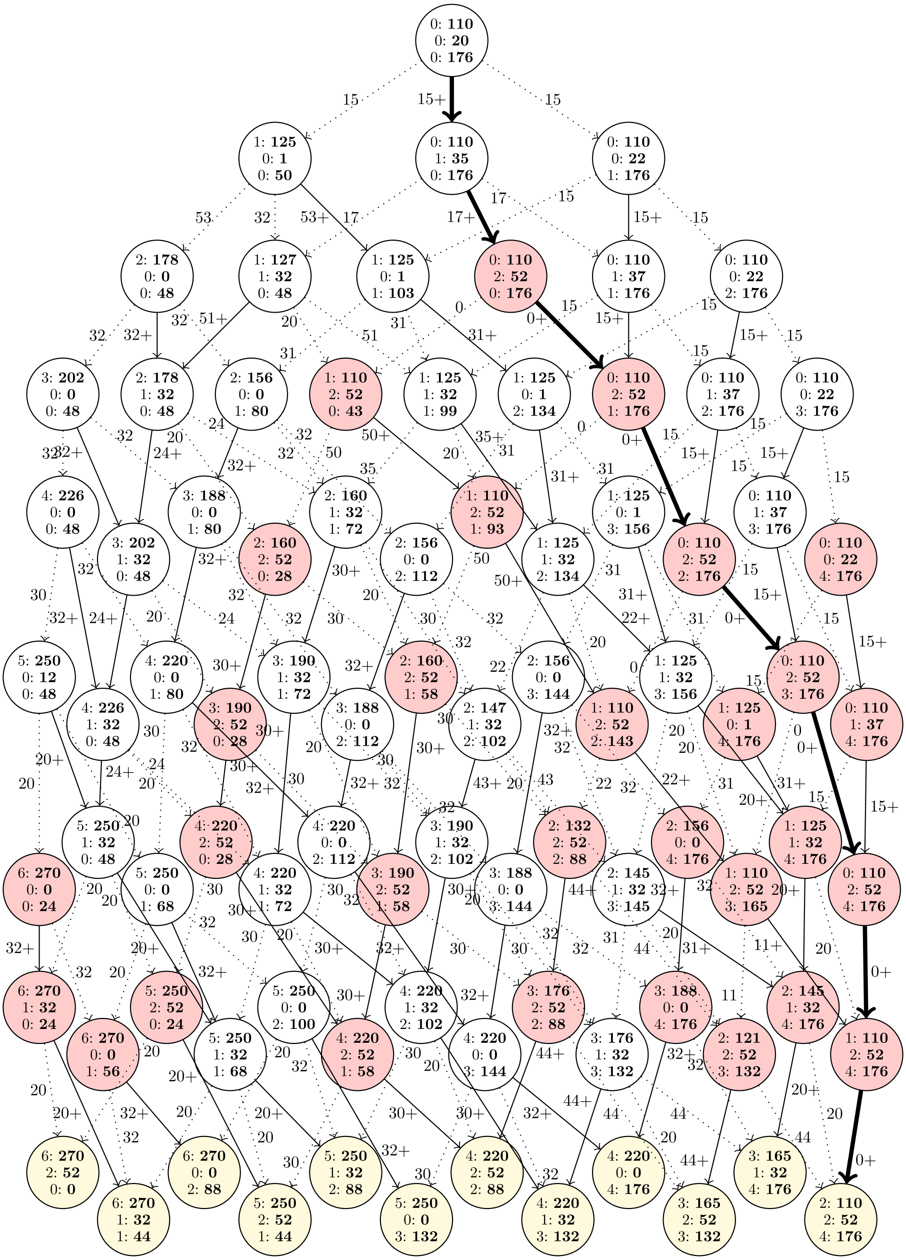


Fig. 14 Non-Monotonic Prices with the `last-in-first-out` Rule

trajectory *for the whole game* is exactly the same. We remark that these observations will play a role when we prove that, for any tie-breaking rule, there is a sequential auction with non-monotonic prices.

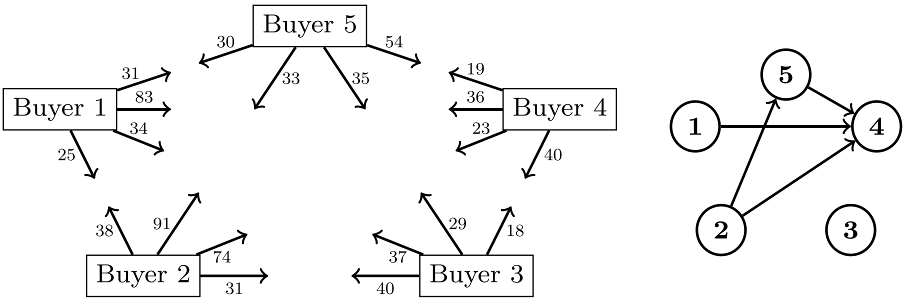


Fig. 15 The Overbidding Graph $G(\{1, 2, 3, 4, 5\}, 40)$

Again, we emphasize that there is nothing inherently perverse about this example. The form of the valuation functions, namely decreasing marginal valuations, is standard. As explained, the equilibrium concept studied is the appropriate one for sequential auctions. Finally, the non-monotonic price trajectory is not the artifact of an aberrant tie-breaking rule; we will now prove that non-monotonic prices are exhibited under any tie-breaking rule.

4 Non-Monotonic Prices under General Tie-Breaking Rules

Next we prove that for any tie-breaking rule there is a sequential auction on which it produces a non-monotonic price trajectory. To do this, we must first formally define the set of all tie-breaking rules.

4.1 Classifying the set of Tie-Breaking Rules

Our definition of the set of tie-breaking rules will utilize the concept of an *overbidding graph*, introduced by [24]. For any price p and any set of bidders S, the overbidding graph $G(S, p)$ contains a labelled vertex for each buyer in S and an arc (i, j) if and only if $v_{i,i} - p > v_{i,j}$. For example, recall the auction with interdependent valuations seen in Fig 8. This is reproduced in Fig. 15 along with its overbidding graph $G(\{1, 2, 3, 4, 5\}, 40)$.

But what does the overbidding graph have to do with tie-breaking rules? First, recall that the drop-out bid β_i is unique for any buyer i, regardless of the tie-breaking rule. Consequently, whilst the tie-breaking rule will also be used to order buyers that are eliminated at prices below the final price p^*, such choices are irrelevant with regards to the final winner. Thus, the only relevant factor is how a decision rule selects a winner from amongst those buyers S^* whose drop-out bids are p^*. Second, recall that at the final price p^* the remaining buyers are eliminated one-by-one until there is a single winner. However, a buyer *cannot* be eliminated if there remains another buyer still in the auction that it wishes to beat at price p^*. That is, buyer i must be eliminated after buyer j if there is an arc (i, j) in the overbidding graph. Thus, the order of eliminations given by the tie-breaking rule must be consistent with the overbidding graph. In particular, the winner can only be selected from amongst the *source vertices*[2] in the overbidding graph $G(S^*, p^*)$. For example, in Fig. 15 the source vertices are $\{1, 2, 3\}$. Note that this explains why the tie-breaking rules `preferential-ordering`, `first-in-first-out` and `last-in-first-out` chose Buyer 1, Buyer 2 and Buyer 3 as winners but none of them selected Buyer 4 or Buyer 5. Observe that the overbidding graph $G(S^*, p^*)$ is *acyclic*; if it contained a directed cycle then the price in the ascending auction would be forced to rise further. Because every directed acyclic graph contains at least one source vertex, any tie-breaking rule does have at least one choice for winner.

Thus a tie-breaking rule is simply a function $\tau : H \rightarrow \sigma(H)$. Here the domain of the function is the set of labelled, directed acyclic graphs and $\sigma(H)$ is the set of source nodes in H. Consequently, two tie-breaking rules are equivalent if they correspond to the same function τ. We are now ready to present our main result.

4.2 Non-Monotonic Prices for Any Tie-Breaking Rule

Theorem 4.1 *For any tie-breaking rule, there is a sequential auction with non-monotonic prices.*

Proof We consider exactly the same example as in Theorem 3.1. That is, we have three buyers and eight items where Buyer 1 has marginal valuations $\{55, 55, 55, 55, 30, 20, 0, 0\}$, Buyer 2 has marginal valuations $\{32, 20, 0, 0, 0, 0, 0, 0\}$, and Buyer 3 has marginal valuations $\{44, 44, 44, 44, 0, 0, 0, 0\}$.

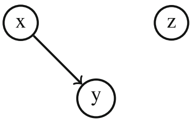
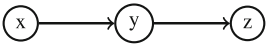
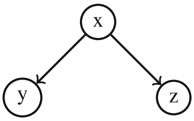
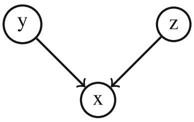
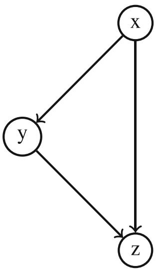
First let's calculate how many tie-breaking rules there are for this auction. To count this we must consider all directed acyclic graphs with labels in $\{1, 2, 3\}$. Note that we must have at least two buyers with drop-out values equal to the final price p^* otherwise the auction would have terminated earlier. Thus it suffices to consider directed acyclic graphs with either two or three vertices. There are 8 such topologies that produce 34 labelled directed acyclic graphs and 12,288 tie-breaking rules! This is all illustrated in Table 1.

Luckily we do not need to examine all these tie-breaking rules separately. It turns out that the set of tie-breaking rules can be partitioned into exactly *ten* classes. Specifically, any tie-breaking rule produces one of just ten possible (in terms of distinct node valuations) extensive forms for this sequential auction. Two of these we have seen before. The first is the extensive form shown in Fig. 12 (and also shown in Fig. 13), that is produced by both `preferential-ordering` and `first-in-first-out`. The second is the extensive form shown in Fig. 14 produced by `last-in-first-out`.

Let's explain why there are only eight other feasible extensive forms. For any tie-breaking rule, as we work up from the sink nodes there are many nodes where the tie-breaking rule is required. Given this fact, why doesn't the total number of distinct extensive forms blow-up multiplicatively? As previously alluded to, when we apply a tie-breaking rule there are two possibilities that arise. In the first possibility,

[2] A *source* is a vertex v with in-degree zero; that is, there no arcs pointing into v.

Table 1 Labelled Directed Acyclic Graphs

Directed Acylic Graph	# Labelled Graphs	# Sources
	3	2
	6	1
	1	3
	6	2
	6	1
	3	1
	3	2
	6	1
Total # Labelled DAGs	34	
Total # Tie-Breaking Rules	$1^{21} \cdot 2^{12} \cdot 3^{1} = 12,288$	

the node valuations are the same regardless of which buyer is selected by the rule. Indeed this is why `preferential-ordering` and `first-in-first-out` can produce the same extensive form. For example, consider the node $(3, 0, 2)$ where Buyer 1 wins with `preferential-ordering` but Buyer 3 wins with `first-in-first-out`; in either case the node valuations are identical, namely $(188, 0, 112)$ as shown in Figs. 12 and 13. For our purpose, such nodes are of no importance.

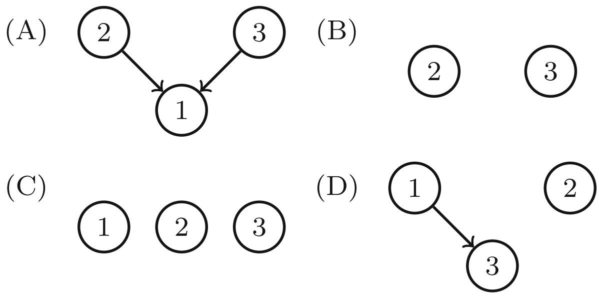


Fig. 16 The Four Critical Overbidding Graphs

In the second possibility, the node valuations do vary depending upon which buyer is selected by the tie-breaking rule. It turns out, however, that of the 34 labelled directed acyclic graphs only 4 of these overbidding graphs affect the extensive form node valuations. These four critical overbidding graphs, which we call A, B, C and D, are shown in Fig. 16.

So why are these these the only four overbidding graphs that matter? The reader may verify that, working upwards from the sink nodes, the first such nodes where the choice of tie-breaking rule matters occur at depth 4. Specifically, at the three nodes $(4, 0, 0)$, $(1, 0, 3)$ and $(0, 1, 3)$. Now the nodes $(1, 0, 3)$ and $(0, 1, 3)$ both correspond to the overbidding graph A whilst the node $(4, 0, 0)$ corresponds to the overbidding graph B. For the overbidding graph A the tie-breaking rule must select either the sink vertex 2 or the sink vertex 3 to win. Moreover, by definition, it must make the same choice at both $(1, 0, 3)$ and $(0, 1, 3)$. Furthermore, regardless of this choice, as we work up the extensive form the nodes $(1, 0, 2)$, $(0, 1, 2)$, $(0, 0, 3)$, $(0, 1, 1)$, $(0, 0, 2)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(0, 0, 0)$ also all have the overbidding graph A and, thus, must also have the same winner.

The choice of winner at $(4, 0, 0)$ for overbidding graph B is also between Buyer 2 and Buyer 3, but in this case, the effect is more subtle. If Buyer 2 wins then the overbidding graph D is induced at node $(3, 0, 0)$, whereas if Buyer 3 wins then the overbidding graph C is induced at $(3, 0, 0)$. In the former case, the overbidding graph D arises at node $(2, 0, 0)$ regardless the choice of winner at $(3, 0, 0)$. In the latter case, there are three possible winners in the overbidding graph C at $(3, 0, 0)$. If Buyer 1 or Buyer 3 win these produce the same node valuations and give the overbidding graph C at $(2, 0, 0)$; if Buyer 2 wins this gives the overbidding graph D at $(2, 0, 0)$. A decision tree showing all the possible choices is shown in Fig. 17. The reader may verify that these are the only decisions that affect the valuations at the nodes. Thus there are ten possible extensive forms, where `Yes/No` details whether or not a monotonic price trajectory is produced. Where the tie-breaking rules `preferential-ordering`, `first-in-first-out`, and `last-in-first-out` fit in this decision tree are highlighted in the figure.

Several observations are in order. First, not all of the classes of tie-breaking rule give non-monotonic price trajectories. An example of a tie-breaking rule with monotonic prices is shown in Fig. 18. In fact, the choices made on the overbidding graphs

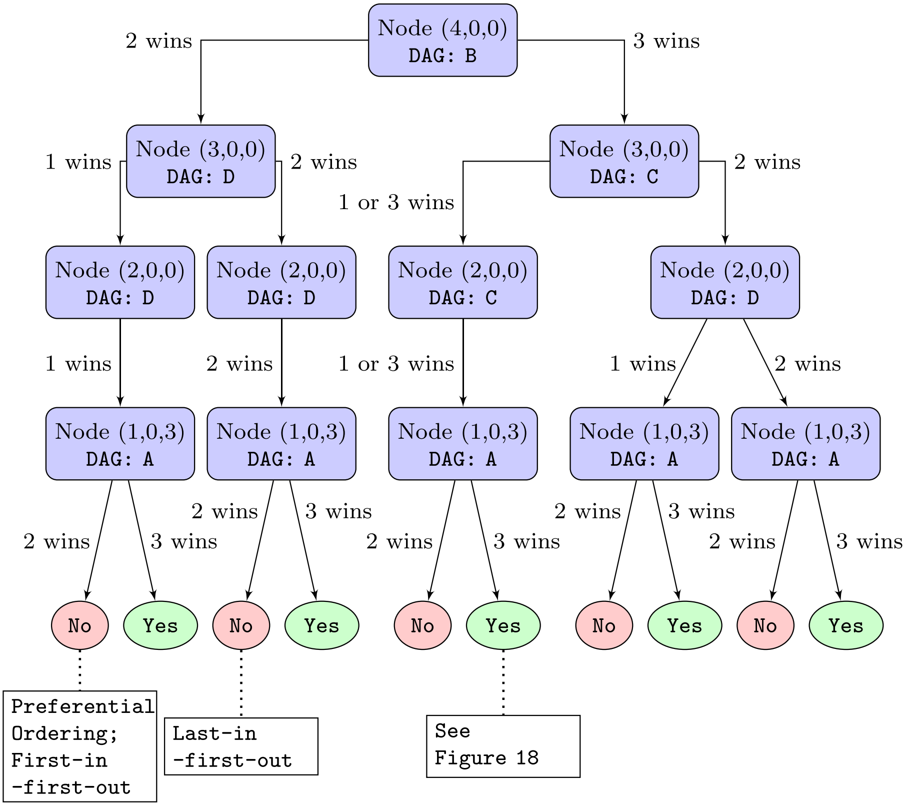


Fig. 17 Monotonic Prices: Yes or No? *A Decision Tree Partitioning the Tie-Breaking Rules into Ten Classes*

B, C and D only affect valuations on nodes off the equilibrium path. The equilibrium path itself is determined uniquely by the choice made for the overbidding graph A. If the winner there is Buyer 2 then the prices are non-monotonic; if the winner there is Buyer 3 then the prices are monotonic.

We are now ready to complete the proof of the theorem. As we have just seen, any tie-breaking rule can be classified into one of ten classes depending upon its choices on this sequential auction. Five of the classes lead to non-monotonic prices on this instance. For the other five classes of tie-breaking rule we need to construct different examples on which they induces non-monotonic prices. But this is easy to do! Take exactly the same example but with the labels of Buyer 2 and Buyer 3 interchanged. The equilibrium paths for this sequential auction using any rule in the other five classes will then have non-monotonic price trajectories.

4.3 Negative Utilities and Overbidding

We now discuss a couple of interesting observations that arise from this specific sequential auction. First we recall another interesting property of two-buyer

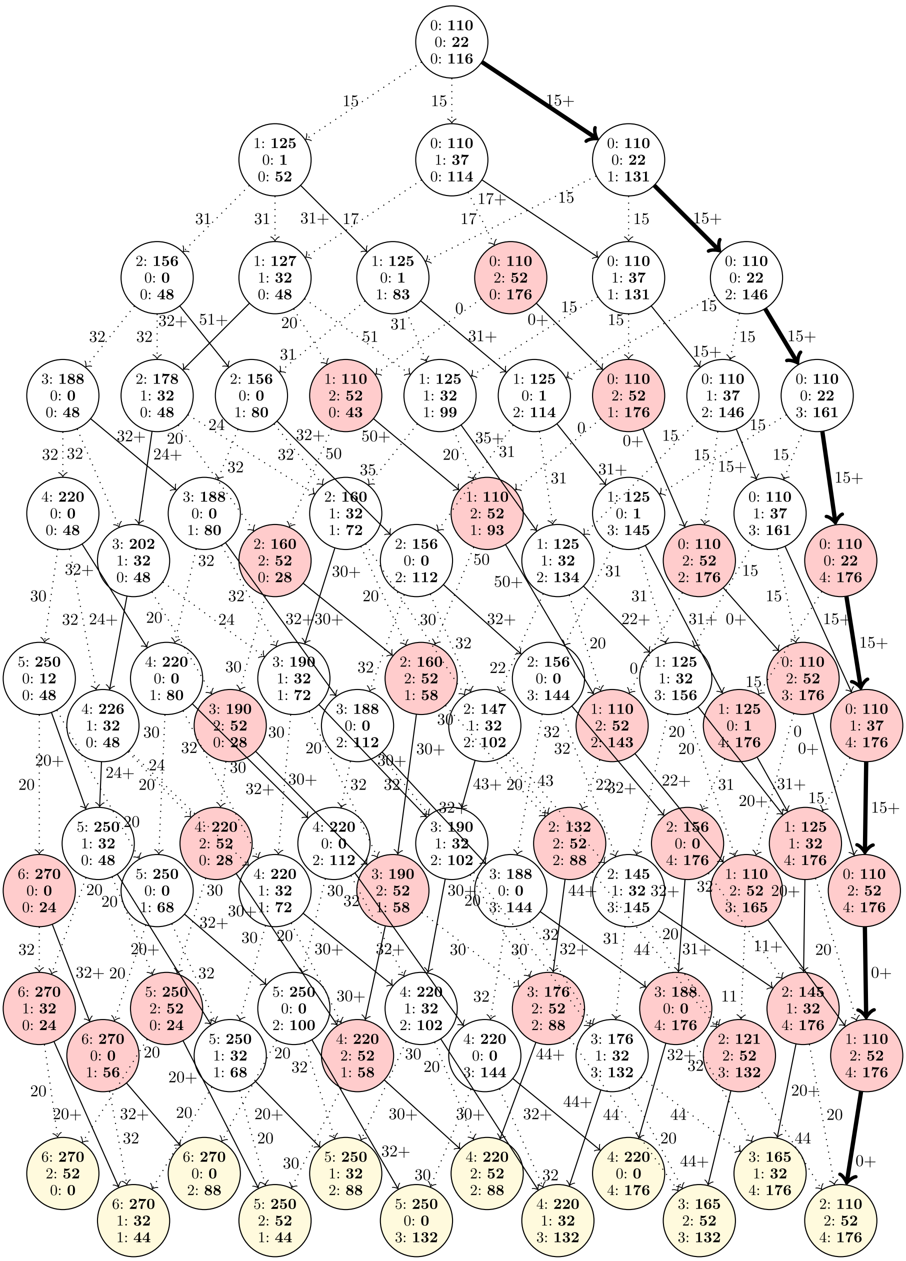


Fig. 18 A Tie-Breaking Rule with Monotonic Prices

sequential auctions: in each round of the auction each buyer has a non-negative value for winning the item over the other agent [12]. Interestingly, this property also fails to hold for multi-buyer sequential auctions!

Theorem 4.2 *There are multi-buyer sequential auctions with weakly decreasing marginal valuations that have subgames where one agent has a negative value for winning against one other agent.*

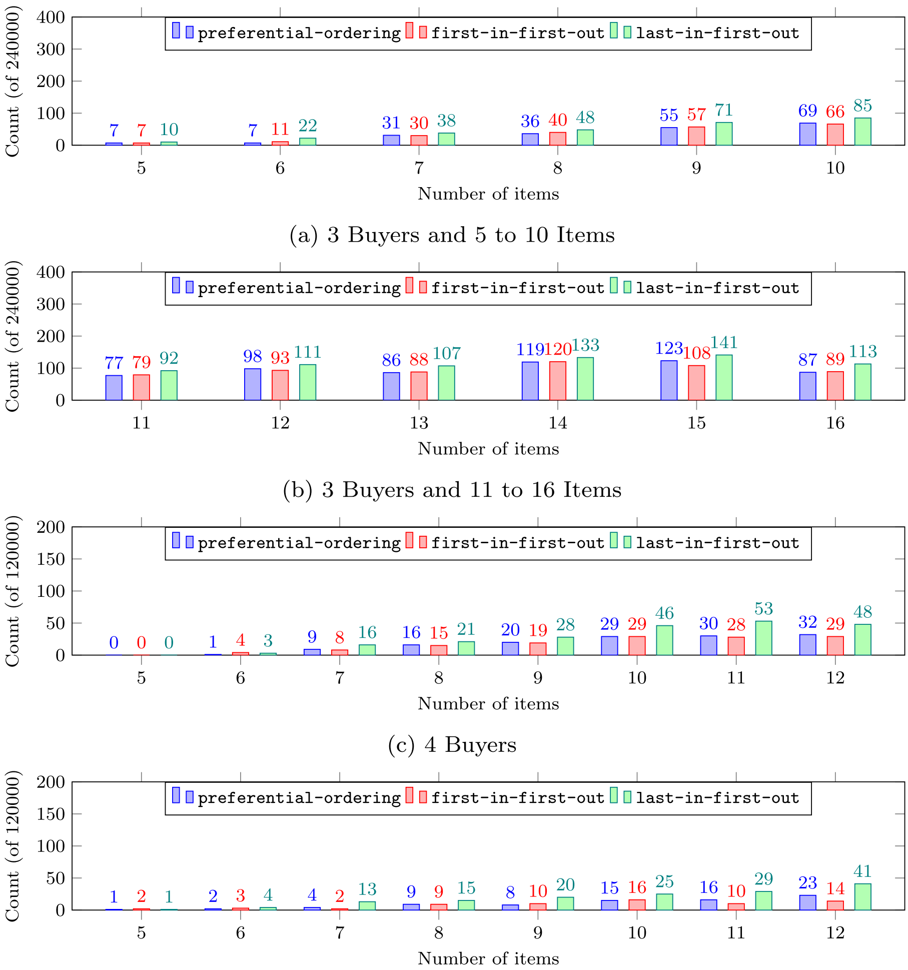
Proof Consider again the sequential auction shown in Fig. 18. Focus upon the auctions with interdependent valuations corresponding to the subgames rooted at the nodes $(0, 1, 0)$, $(0, 1, 1)$ and $(0, 1, 2)$. In all three cases, Buyer 3 has a negative value from winning over Buyer 2. For example, at node $(0, 1, 0)$ Buyer 3 has a utility of 131 for winning but a utility of 176 if Buyer 2 wins. (Note that Buyer 3 does have a positive value for defeating Buyer 1, specifically $131 - 48 = 83$.) Of course, this also implies there are sequential auctions with weakly decreasing marginal valuation functions where one agent has a negative value for winning the *first item* over one other agent.

Second, observe in Fig. 12 (see also Fig. 11) that in the first round Buyer 3 has a value of $176-66 = 110$ for winning over Buyer 1. This far exceeds its marginal value of 44 for obtaining one item. A similar property can be seen in Figs. 13 and 14. Such "overbidding" also arises in two-buyer sequential auctions. The reader may wonder, however, whether this type of "overbidding" is responsible for the generation of non-monotonic price trajectories in multi-buyer auctions. This is not the case. To verify this we repeated all six million experiments described in Section 5 with the ascending price mechanism modified to exclude the possibility of a buyer bidding higher than their marginal value for their next unit of the good. The proportion of instances with non-monotonic price trajectories was similar (roughly 10% less). Moreover, there are instances where such "overbidding" does not arise but where the prices are non-monotonic.

5 Experiments

Our experiments were based on a dataset of over six million multi-buyer sequential auctions with non-increasing valuation functions randomly generated from different natural discrete probability distributions. Our goal was to observe the proportion of non-monotonic price trajectories in these sequential auctions and to see how this varied with (i) the number of buyers, (ii) the number of items, (iii) the distribution of valuation functions, and (iv) the tie-breaking rule. To do this, for each auction, we computed the subgame perfect equilibrium corresponding to the dropout bids and evaluated the prices along the equilibrium path to test for non-monotonicity.

We repeated this test for each of the three tie breaking rules described in Section 2.4.1, namely `preferential-ordering`, `first-in-first-out` and `last-in-first-out`. The results from our 6,240,000 randomly generated sequential auctions are shown in Fig. 19.



(a) 3 Buyers and 5 to 10 Items

(b) 3 Buyers and 11 to 16 Items

(c) 4 Buyers

(d) 5 Buyers

Fig. 19 Bar Charts showing the Frequency of Non-Monotonic Price Trajectories

5.1 Dataset Generation

We now describe the methods used to generate our auction dataset. Our generator was implemented in `C++11`, using the GNU Compiler Collection (version 5.1.0) and the standard random number library. The random number library provides classes that generate pseudo-random numbers. These classes include both uniform random bit generators (URBGs), which generate integer sequences with a uniform distribution, and random number distributions, which convert the output of a URBG into various statistical distributions (such as uniform, binomial or Poisson distributions). In our experiments, we used the MT19937-64 implementation of the widely-used Mersenne Twister URBG [20] along with the standard `uniform_int_distribution`, `binomial_distribution` and `poisson_distribution` classes to generate the valuation functions in the dataset. We restricted our attention to integral non-increasing marginal valuations and bounded the maximum marginal value of a single item by 100. The purpose of this choice of constraints was to allow for a wide variety of auction instances whilst still allowing for a reasonable chance for ties to arise in the ascending auction mechanism, thus enabling us to observe any potential effects of varying the tie-breaking rule.

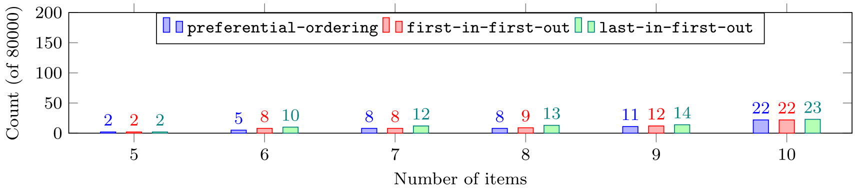
Our dataset contains auctions with $n = 3, 4$ and 5 different buyers. For the 3-buyer case, we varied the number T of items from $T = 2$ to $T = 16$. For each auction size, we generated a total of 240,000 instances. Specifically, let V_i be the valuation function of buyer i, that is, $V_i(0) = 0$ and V_i assigns a non-negative value for every integer ℓ, $1 \leq \ell \leq T$, corresponding to buyer i's value for any set of ℓ items. Let $v_i(\ell)$ be the marginal value of buyer i for winning an ℓ^{th} item, that is, $v_i(\ell) = V_i(\ell) - V_i(\ell - 1)$. We used the three aforementioned distributions to each generate 80,000 sets of valuation functions. To generate the values $v_i(\ell)$, for each buyer i we first uniformly selected a maximum marginal value w_i in the interval $[1, 100]$. For half the instances, we generated w_i independently for every player, and for the other half, we made w_i equal for all players. Subsequently, for each buyer i, we chose the number of nonzero valuations m_i uniformly in $[1, T]$. For the first distribution, we then independently generated and sorted m_i values uniformly in $[1, w_i]$, and padded this sequence with $T - m_i$ zeros, to generate a decreasing integer sequence: the valuation function for buyer i. For the second distribution, for each buyer i we generated m_i values from a binomial distribution with parameters $n = w_i$ and $p = 0.5$, and sorted and padded this sequence with $T - m_i$ zeros. For the third distribution, we let $u_{i,1} = w_i$, and for each $j \geq 2$ we let $u_{i,j} = \max(0, u_{i,j-1} - x_j)$, where x_j was drawn from a Poisson distribution with parameter $\lambda = \frac{w_i}{m_i}$.

The above steps were repeated for the 4-buyer and 5-buyer cases, varying the number of items from $T = 2$ to $T = 12$ in each case. In each of these cases, we generated a total of 120,000 instances for each auction size, with 40,000 instances generated from each of the three distributions.

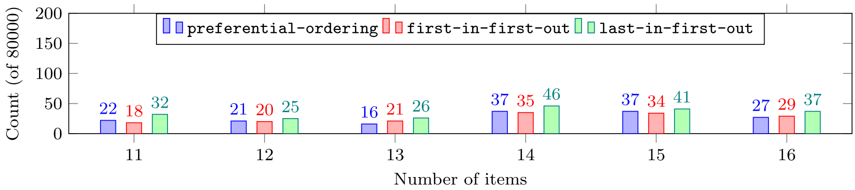
Let us comment on the reasoning behind these choices for the sizes of our auctions. Our sequential auctions, with at most five bidders and at most sixteen items, could be solved extremely quickly; this allowed us to analyze our large dataset. But it can be shown that the number of nodes in the extensive form for a sequential auction with n buyers and T items is exactly

$$\sum_{t=0}^{T} \binom{t+n-1}{n-1} = \binom{T+n}{n} = \binom{T+n}{T}$$

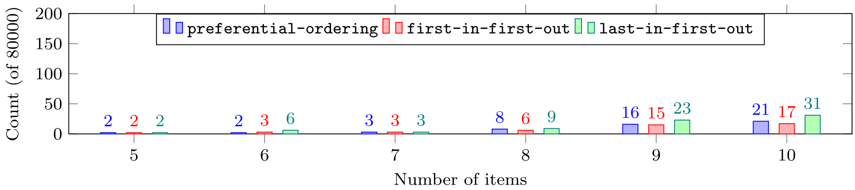
Thus the size of the extensive form grows exponentially in the number of buyers and the number of items. So, whilst with additional time we can easily solve slightly larger instances, we cannot expect to solve significantly larger instances. We remark that the auction sizes we can solve are comparable to many of the real sequential auctions described in the introduction.



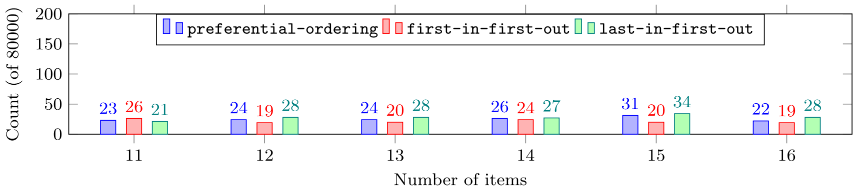
(a) 3 Buyers, Uniform Distribution, 5 to 10 Items



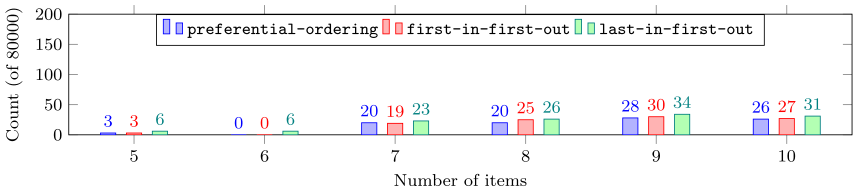
(b) 3 Buyers, Uniform Distribution, 11 to 16 Items



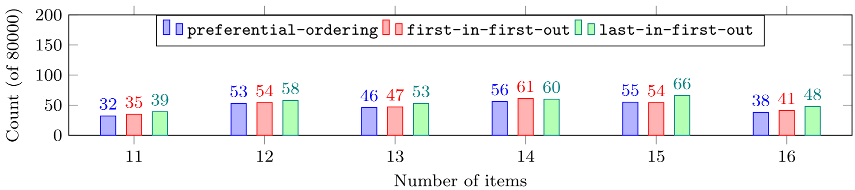
(c) 3 Buyers, Poisson Step-size Distribution, 5 to 10 Items



(d) 3 Buyers, Poisson Step-size Distribution, 11 to 16 Items

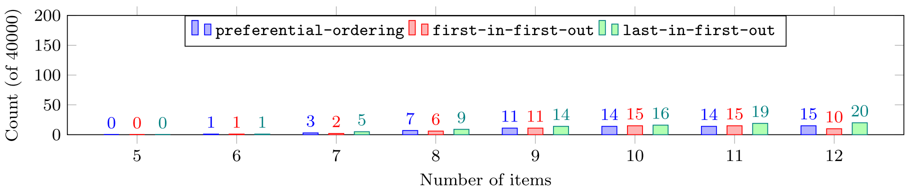


(e) 3 Buyers, Binomial Distribution, 5 to 10 Items

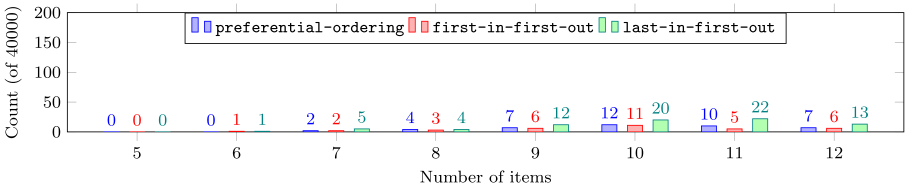


(f) 3 Buyers, Binomial Distribution, 11 to 16 Items

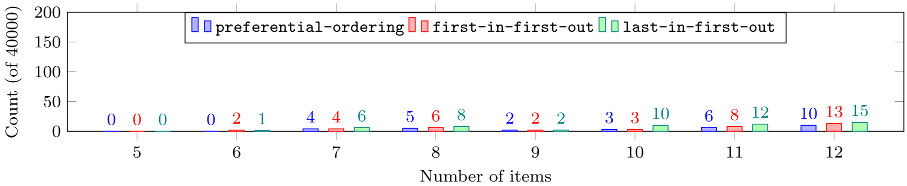
Fig. 20 Frequency of Non-Monotonic Price Trajectories in 3-Buyer Auctions



(a) 4 buyers, uniform distribution



(b) 4 buyers, Poisson step-size distribution



(c) 4 buyers, binomial distribution

Fig. 21 Frequency of Non-Monotonic Price Trajectories in 4-Buyer Auctions

5.2 Experimental Results

The results from our 6,240,000 randomly generated sequential auctions are shown in Fig. 19. In these bar charts there is one bar for each combination of auction size and data structure (`preferential-ordering`, `first-in-firstout` and `last-in-first-out`). Each bar shows the number of auctions of that type that induced non-monotonic prices. For example, for sequential auctions with three buyers and five items that use the preferential-ordering tie-breaking rule, there were 7 auctions out of 240,000 that had non-monotonic prices. For four and five buyers there were 120,000 auctions of each type.

Fig. 19 shows all the tests together. Recall that the valuation functions in each sequential auction were generated in one of three different ways (uniform, Poisson, binomial). In Fig. 20 we show these three cases for 3-buyer sequential auctions. In Figs. 21 and 22 we show them for 4-buyer and 5-buyer auctions respectively. We found no examples with less than 5 items that showed non-monotonicity, so the cases $T = 2, 3, 4$ are omitted.

As can be observed, for a fixed number of buyers, there is a slight upward drift in the proportion of non-monotonic price trajectories as the number of items increases. Intuitively that seems unsurprising, as with longer price sequences there are more

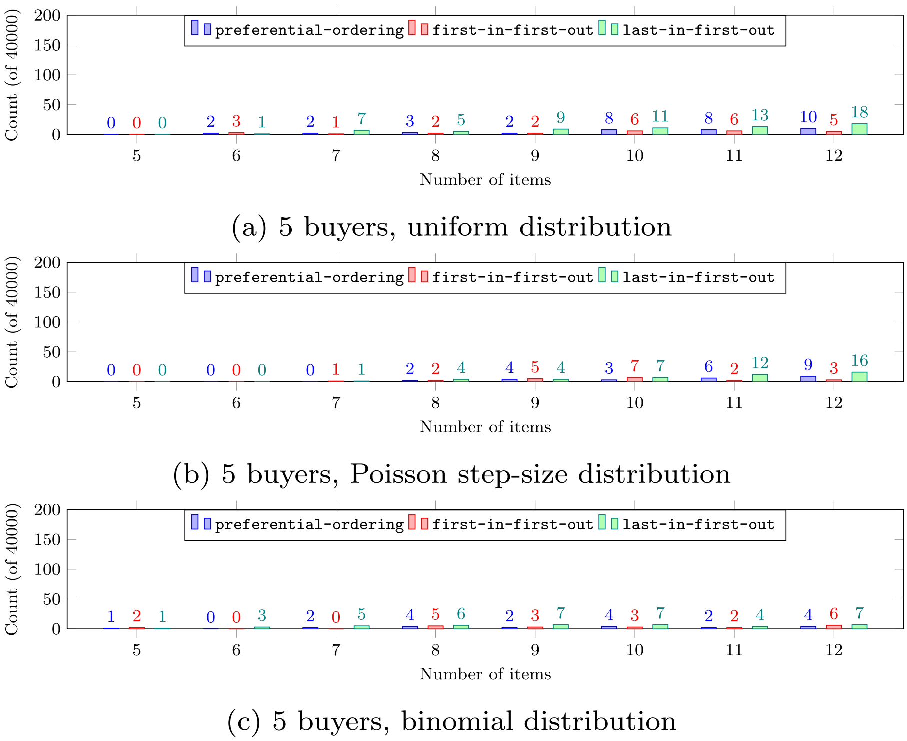


Fig. 22 Frequency of Non-Monotonic Price Trajectories in 5-Buyer Auctions

time periods at which deviations from monotonicity can arise. A very interesting question would be to study the limit of the proportion of non-monotonic price trajectories as the number of items gets very large. Unfortunately, due to the exponential explosion in the number of game tree nodes discussed above, this is a question that cannot be studied experimentally. The main conclusion to be drawn from these experiments is that non-monotonic prices are extremely rare. On the 6,240,000 auction instances, the `preferential-ordering` tie-breaking rule produced just 1,100 violations of the declining price anomaly. The `first-in-first-out` rule gave 986 violations and the `last-in-first-out` rule gave 1,334 violations. The overall observed rate of non-monotonicity over these 18 million tests was 0.000183.

6 Conclusion

In this work, we proved that the declining price anomaly is not guaranteed to hold in the equilibria of full-information sequential auctions with three or more buyers. This result applies to both first-price and second-price sequential auctions. Moreover, it applies regardless of the tie-breaking rule used to generate equilibria in these sequential auctions. To prove this result we presented a refined treatment of subgame perfect equilibria that survive the iterative deletion of weakly dominated strategies. We also experimentally generated a large number of random sequential auction instances and show that non-monotonic price trajectories are extremely rare. One potential direction for future work is to relate the number of agents to the minimum number of items necessary to exhibit a non-monotonic price trajectory at equilibrium. Additionally, our insights into the structure of equilibria in sequential auctions may lead to improvements in other areas of investigation into the properties of these auctions. For instance, it is known [1] that with two buyers, the price of anarchy of identical-item sequential auctions is bounded by $1 - \frac{1}{e}$, but no non-trivial bounds are known for the case of three or more buyers and identical items.

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