



The Declining Price Anomaly Is Not Universal in Multi-buyer Sequential Auctions (But Almost Is)

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Abstract. The declining price anomaly states that the price weakly decreases when multiple copies of an item are sold sequentially over time. The anomaly has been observed in a plethora of practical applications. On the theoretical side, Gale and Stegeman [10] proved that the anomaly is guaranteed to hold in full information sequential auctions with exactly two buyers. We prove that the declining price anomaly is *not* guaranteed in full information sequential auctions with three or more buyers. This result applies to both first-price and second-price sequential auctions. Moreover, it applies regardless of the tie-breaking rule used to generate equilibria in these sequential auctions. To prove this result we provide a refined treatment of subgame perfect equilibria that survive the iterative deletion of weakly dominated strategies and use this framework to experimentally generate a very large number of random sequential auction instances. In particular, our experiments produce an instance with three bidders and eight items that, for a specific tie-breaking rule, induces a non-monotonic price trajectory. Theoretical analyses are then applied to show that this instance can be used to prove that for every possible tie-breaking rule there is a sequential auction on which it induces a non-monotonic price trajectory. On the other hand, our experiments show that non-monotonic price trajectories are extremely rare. In over six million experiments only a 0.000183 proportion of the instances violated the declining price anomaly.

1 Introduction

In a sequential auction identical copies of an item are sold over time. In a private values model with *unit-demand*, risk neutral buyers, Milgrom and Weber [19, 26] showed that the sequence of prices forms a martingale. In particular, expected prices are constant over time.¹ In contrast, on attending a wine auction, Ashenfelter [1] made the surprising observation that prices for identical lots declined over time: “The law of the one price was repealed and no one even seemed to notice!” This *declining price anomaly* was also noted in sequential auctions for

¹ If the values are affiliated then prices can have an upwards drift.

the disparate examples of livestock (Buccola [7]), Picasso prints (Pesando and Shum [21]) and satellite transponder leases (Milgrom and Weber [19]). Indeed, the possibility of decreasing prices in a sequential auction was raised by Sosnick [23] nearly sixty years ago. In the case of wine auctions, proposed causes include absentee buyers utilizing non-optimal bidding strategies (Ginsburgh [11]) and the *buyer's option rule* where the auctioneer may allow the buyer of the first lot to make additional purchases at the same price (Black and de Meza [6]). Minor non-homogeneities amongst the items can also lead to falling prices. For example, in the case of art prints the items may suffer slight imperfections or wear-and-tear, and the auctioneer may sell the prints in decreasing order of quality (Pesando and Shum [21]). More generally, a decreasing price trajectory may arise due to risk-aversion, such as non-decreasing, absolute risk-aversion (McAfee and Vincent [17]) or aversion to price-risk (Mezzetti [18]); see also Hu and Zou [13]. Further potential economic and behavioural explanations have been provided in [2, 11, 25]. Of course, most of these explanations are context-specific. However, in practice the anomaly is ubiquitous: it has now been observed in sequential auctions for, among several other things, antiques (Ginsburgh and van Ours [12]), commercial real estate (Lusht [16]), flowers (van den Berg et al. [5]), fur (Lambson and Thurston [15]), jewellery (Chanel et al. [8]), paintings (Beggs and Graddy [4]) and stamps (Thiel and Petry [24]).

Given the plethora of examples, the question arises as whether this property is actually an anomaly. In groundbreaking work, Gale and Stegeman [10] proved that it is *not* in sequential auctions with *two* bidders. Specifically, in *second-price* sequential auctions with two multiunit-demand buyers, prices are weakly decreasing over time at the unique subgame perfect equilibrium that survives the iterative deletion of weakly dominated strategies. This result applies regardless of the valuation functions of the buyers, and also extends to the corresponding equilibrium in *first-price* sequential auctions. It is worth highlighting that Gale and Stegeman consider multiunit-demand buyers whereas prior theoretical work had focused on the simpler setting of unit-demand buyers. As well as being of more practical relevance (see the many examples above), multiunit-demand buyers can implement more sophisticated bidding strategies. Therefore, it is not unreasonable that equilibria in multiunit-demand setting may possess more interesting properties than equilibria in the unit-demand setting. The restriction to full information in [10] is extremely useful here as it separates away informational aspects and allows one to focus on the strategic properties caused purely by the sequential sales of items and not by a lack of information.

1.1 Results and Overview of the Paper

The result of Gale and Stegeman [10] prompts the question of whether or not the declining price anomaly is guaranteed to hold in general, that is, in sequential auctions with more than two buyers. We answer this question in the negative by exhibiting a sequential auction with three buyers and eight items where prices initially rise and then fall. In order to run our experiments that find this counterexample (to the conjecture that prices are weakly decreasing for multi-buyer sequential auctions) we study in detail the form of equilibria in sequential auctions.

First, it is important to note that there is a fundamental distinction between sequential auctions with two buyers and sequential auctions with three or more buyers. In the former case, each subgame reduces to a standard *auction with independent valuations*. In contrast, in a multi-buyer sequential auction each subgame reduces to an *auction with interdependent valuations*. We present these models in Sects. 2.1 and 2.2. Consequently to study multi-buyer sequential auctions we must study the equilibria of auctions with interdependent valuations. A theory of such equilibria was recently developed by Paes Leme et al. [20] via a correspondence with an ascending price mechanism. In particular, as we discuss in Sect. 2.3, this ascending price mechanism outputs a unique bid value, called the *dropout bid* β_i , for each buyer i . For first-price auctions it is known [20] that these dropout bids form a subgame perfect equilibrium and, moreover, the interval $[0, \beta_i]$ is the exact set of bids that survives *all* processes consisting of the iterative deletion of strategies that are weakly dominated. In contrast, we show that for second-price auctions it may be the case that no bids survive the iterative deletion of weakly dominated strategies; however, we prove in Sect. 2.3 that the interval $[0, \beta_i]$ is the exact set of bids for any losing buyer that survives *all* processes consisting of the iterative deletion of strategies that are weakly dominated *by a lower bid*.

In Sect. 3 we describe the counter-example. We emphasize that the form of the valuation functions used for the buyers are standard, namely, weakly decreasing marginal valuations. Furthermore, the non-monotonic price trajectory does not arise because of the use of an artificial tie-breaking rule; the three most natural tie-breaking rules, see Sect. 2.4, all induce the same non-monotonic price trajectory. Indeed, we present an even stronger result in Sect. 4: for *any* tie-breaking rule, there is a sequential auction on which it induces a non-monotonic price trajectory. This lack of weakly decreasing prices provides an explanation for why multi-buyer sequential auctions have been hard to analyze quantitatively. We provide a second explanation in the full paper, where we present a three-buyer sequential auction that does satisfy weakly decreasing prices but which has subgames where some agent has a negative value from winning against one of the two other agents. Again, this contrasts with the two-buyer case where every agent always has a non-negative value from winning against the other agent in every subgame.

Finally in Sect. 5, we describe the results obtained via our large scale experimentations. These results show that whilst the declining price anomaly is not universal, exceptions are extremely rare. Specifically, from a randomly generated dataset of over six million sequential auctions only a 0.000183 proportion of the instances produced non-monotonic price trajectories. Consequently, these experiments are consistent with the practical examples discussed in the introduction. Of course, it is perhaps unreasonable to assume that subgame equilibria arise in practice; we remark, though, that the use of simple bidding algorithms by bidders may also lead to weakly decreasing prices in a multi-buyer sequential auction. For example, Rodriguez [22] presents a method called the *residual monopsonist procedure* inducing this property in restricted settings.

2 The Sequential Auction Model

Here we present the full information sequential auction model. There are T identical items and n buyers. Exactly one item is sold in each time period over T time periods. Buyer i has a value $V_i(k)$ for winning exactly k items. Thus $V_i(k) = \sum_{\ell=1}^k v_i(\ell)$, where $v_i(\ell)$ is the marginal value buyer i has for an ℓ th item. This induces an extensive form game. To analyze this game it is informative to begin by considering the 2-buyer case studied by Gale and Stegeman [10].

2.1 The Two-Buyer Case

During the auction, the relevant history is the number of items each buyer has currently won. Thus we may compactly represent the extensive form (“tree”) of the auction using a directed graph with a node (x_1, x_2) for any pair of non-negative integers that satisfies $x_1 + x_2 \leq T$. The node (x_1, x_2) induces a subgame with $T - x_1 - x_2$ items for sale and where each buyer i already possesses x_i items. Note there is a *source node*, $(0, 0)$, corresponding to the whole game, and *sink nodes* (x_1, x_2) , where $x_1 + x_2 = T$. The values Buyer 1 and Buyer 2 have for a sink node (x_1, x_2) are $\Pi_1(x_1, x_2) = V_1(x_1)$ and $\Pi_2(x_1, x_2) = V_2(x_2)$, respectively. Take a node (x_1, x_2) , where $x_1 + x_2 = T - 1$. This node corresponds to the final round of the auction, where the last item is sold, and has directed arcs to the sink nodes $(x_1 + 1, x_2)$ and $(x_1, x_2 + 1)$. For the case of second-price auctions, it is then a weakly dominant strategy for Buyer 1 to bid its marginal value $v_1(x_1 + 1) = V_1(x_1 + 1) - V_1(x_1)$; similarly for Buyer 2. Of course, this marginal value is just $v_1(x_1 + 1) = \Pi_1(x_1 + 1, x_2) - \Pi_1(x_1, x_2 + 1)$, the difference in value between winning and losing the final item. If Buyer 1 is the highest bidder at (x_1, x_2) , that is, $\Pi_1(x_1 + 1, x_2) - \Pi_1(x_1, x_2 + 1) \geq \Pi_2(x_1, x_2 + 1) - \Pi_2(x_1 + 1, x_2)$, then we have that

$$\begin{aligned} \Pi_1(x_1, x_2) &= \Pi_1(x_1 + 1, x_2) - (\Pi_2(x_1, x_2 + 1) - \Pi_2(x_1 + 1, x_2)) \\ \Pi_2(x_1, x_2) &= \Pi_2(x_1 + 1, x_2) \end{aligned}$$

Symmetric formulas apply if Buyer 2 is the highest bidder. Hence we may recursively define a value for each buyer for each node. The iterative elimination of weakly dominated strategies leads to a subgame perfect equilibrium [3, 10].

Example: Consider a two-buyer sequential auction with two items, where the marginal valuations are $\{v_1(1), v_1(2)\} = \{10, 8\}$ and $\{v_2(1), v_2(2)\} = \{6, 3\}$. This game is illustrated in Fig. 1. The base case with the values of the sink nodes is shown in Fig. 1(a). The first row in each node refers to Buyer 1 and shows the number of items won (in plain text) and the corresponding value (in bold); the second row refers to Buyer 2. The outcome of the second-price sequential auction, solved recursively, is then shown in Fig. 1(b). Arcs are labelled by the bid value; here arcs for Buyer 1 point left and arcs for Buyer 2 point right. Solid arcs represent winning bids and dotted arcs are losing bids. The equilibrium path is shown in bold. Figure 1(c) shows the corresponding first-price auction, where we make the standard assumption of a fixed small bidding increment, and the notation p^+ and p

are respectively used to denote a winning bid of value p and a losing bid equal to the maximum value smaller than p . For simplicity, all the figures we present in the rest of the paper will be for first-price auctions; equivalent figures can be drawn for the case of second-price auctions. Observe that this example exhibits the *declining price anomaly*: in the equilibrium, the first item has price 5 and the second item has price 3. As stated, Gale and Stegeman [10] showed that this example is not an exception.

Theorem 1 [10]. *In a 2-buyer second-price sequential auction there is a unique equilibrium that survives the iterative deletion of weakly dominated strategies. Moreover, at this equilibrium prices are weakly declining.* □

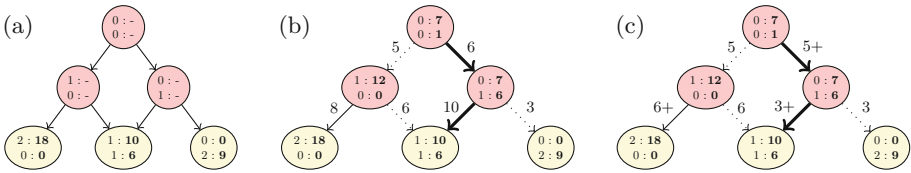


Fig. 1. Sequential auction examples

We remark that the subgame perfect equilibrium that survives iterative elimination is unique in terms of the values at the nodes. Moreover, given a fixed tie-breaking rule, the subgame perfect equilibrium also has a unique equilibrium path in each subgame. In addition, Theorem 1 also applies to first-price sequential auctions. The question of whether or not it applies to sequential auctions with more than two buyers remained open. We resolve this question in the rest of this paper. To do this, let's first study equilibria in the full information sequential auction model when there are more than two buyers.

2.2 The Multi-buyer Case

The underlying model of [10] extends simply to sequential auctions with $n \geq 3$ buyers. There is a node (x_1, x_2, \dots, x_n) for each set of non-negative integers satisfying $\sum_{i=1}^n x_i \leq T$. There is a directed arc from (x_1, x_2, \dots, x_n) to $(x_1, x_2, \dots, x_{j-1}, x_j + 1, x_{j+1}, \dots, x_n)$ for each $1 \leq j \leq n$. Thus each non-sink node has n out-going arcs. This is problematic: whilst in the final time period each buyer has a value for winning and a value for losing, this is no longer the case recursively in earlier time periods. Specifically, buyer i has a value for winning, but $n - 1$ (different) values for losing depending upon the identity of the buyer $j \neq i$ who wins. Thus each node in the multi-buyer case corresponds to an *auction with interdependent valuations*. Formally, this is a single-item auction where each buyer i has a value $v_{i,i}$ for winning the item and a value $v_{i,j}$ if buyer j wins the item, for each $j \neq i$. These auctions, also called *auctions with externalities*, were introduced by Funk [9] and by Jehiel

and Moldovanu [14]. Their motivations were applications where losing participants were not indifferent to the identity of the winner; examples include firms seeking to purchase a patented innovation, take-over acquisitions of a smaller company in an oligopolistic market, and sports teams competing to sign a star athlete. Therefore to understand multi-buyer sequential auctions we must first understand equilibria in auctions with interdependent valuations. This is not a simple task; indeed, such an understanding was only recently provided by Paes Leme et al. [20].

2.3 Equilibria in Auctions with Interdependent Valuations

We can explain the result of [20] via an ascending price auction. Imagine a two-buyer ascending price auction where the valuations of the buyers are $v_1 > v_2$. The requested price p starts at zero and continues to rise until the point where the second buyer drops out. Of course, this happens when the price reaches v_2 , and so Buyer 1 wins for a payment $p^+ = v_2$, which is exactly the outcome expected from a first-price auction. To generalize this to multi-buyer settings we can view this process as follows. At a price p , buyer i remains in the auction as long as there is at least one buyer j still in the auction who buyer i is willing to pay a price p to beat; that is, $v_{i,i} - p > v_{i,j}$. The last buyer to drop out wins at the corresponding price. Even in this setting, this procedure produces a unique *dropout bid* β_i for each buyer i , as illustrated in Fig. 2. In these diagrams the label of an arc from buyer i to buyer j is $w_{i,j} = v_{i,i} - v_{i,j}$. That is, buyer i is willing to pay up to $w_{i,j}$ to win if the alternative is that buyer j wins the item. Now consider running our ascending price procedure for these auctions. In Fig. 2(a), Buyer 1 drops out when the price reaches 18. Since Buyer 1 is no longer active, Buyer 4 drops out at 23. Buyer 3 wins when Buyer 2 drops out at 31. Thus the drop-out bid of Buyer 3 is 31^+ . Observe that Buyer 2 loses despite having very high values for winning against Buyer 1 and Buyer 4. The example of Fig. 2(b) is more subtle. Here Buyer 2 drops out at price 24. But Buyer 3 only wanted to beat Buyer 2 at this price so it then immediately drops out at the same price. Now Buyer 1 only wanted to beat Buyer 2 and Buyer 3 at this price, so it then immediately drops out at the same price. This leaves Buyer 4 the winner at price 24^+ .

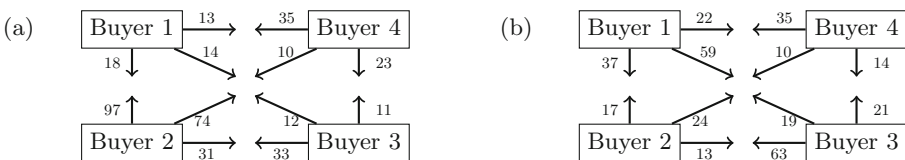


Fig. 2. DROP-OUT BID EXAMPLES. In these two examples the dropout bid vectors $(\beta_1, \beta_2, \beta_3, \beta_4)$ are $(18, 31, 31^+, 23)$ and $(24, 24, 24, 24^+)$, respectively.

As well as being solutions to the ascending price auction, the dropout bids have a much stronger property that makes them the natural and robust prediction for auctions with interdependent valuations. Specifically, Paes Leme et al. [20] proved that, for each buyer i , the interval $[0, \beta_i]$ is the set of strategies that survive *any sequence* consisting of the iterative deletion of weakly dominated strategies. This is formalized as follows. Take an n -buyer game with strategy sets S_1, S_2, \dots, S_n and utility functions $u_i : S_1 \times S_2 \times \dots \times S_n \rightarrow \mathbb{R}$. Then $\{S_i^\tau\}_{i,\tau}$ is a *valid sequence* for the iterative deletion of weakly dominated strategies if for each τ there is a buyer i such that (i) $S_j^\tau = S_j^{\tau-1}$ for each buyer $j \neq i$ and (ii) $S_i^\tau \subset S_i^{\tau-1}$ where for each strategy $s_i \in S_i^{\tau-1} \setminus S_i^\tau$ there is an $\hat{s}_i \in S_i^\tau$ such that $u_i(\hat{s}_i, s_{-i}) \geq u_i(s_i, s_{-i})$ for all $s_{-i} \in \prod_{j:j \neq i} S_j^\tau$, and with strict inequality for at least one s_{-i} . We say that a strategy s_i for buyer i *survives* the iterative deletion if for any valid sequence $\{S_i^\tau\}_{i,\tau}$ we have $s_i \in \bigcap_\tau S_i^\tau$.

Theorem 2 [20]. *Given a first-price auction with interdependent valuations, for each buyer i , the set of bids that survive the iterative deletion of weakly dominated strategies is exactly $[0, \beta_i]$.* □

An exact analogue of Theorem 2 does *not* hold for second-price auctions with interdependent valuations. Indeed, there exist examples in which the set of strategies that survive iterative deletion is empty. However, consideration of these example shows that the problem occurs when a strategy is deleted because it is weakly dominated by a *higher* value bid. Observe that this can never happen for a potentially winning bid. Thus Theorem 2 still holds in first-price auctions when we restrict attention to sequences consisting of the iterative deletion of strategies that are weakly dominated by a *lower* bid. We can also show that the corresponding theorem holds for second-price auctions. The full technical details of the proof are deferred to the full paper.

Theorem 3. *Given a second-price auction with interdependent valuations, for each losing buyer i , the set of bids that survive the iterative deletion of strategies that are weakly dominated by a lower bid is exactly $[0, \beta_i]$.* □

We are now almost ready to be able to find equilibria in the sequential auction experiments we will conduct. This, in turn, will allow us to present a sequential auction with non-monotonic prices. Before doing so, one final factor remains to be discussed regarding the transition from equilibria in auctions with interdependent valuations to equilibria in sequential auctions.

2.4 Equilibria in Sequential Auctions: Tie-Breaking Rules

As stated, the dropout bid of each buyer is uniquely defined. However, our description of the ascending auction may leave some flexibility in the choice of winner. Specifically, it may be the case that simultaneously more than one buyer wishes to drop out of the auction. If this happens at the end of the ascending price procedure then any of these buyers could be selected as the winner. To fully define the ascending auction we must incorporate a tie-breaking rule to order the buyers when more

than one wish to drop out simultaneously. In an auction with interdependent valuations the tie-breaking rule only affects the choice of winner, but otherwise has no structural significance. However, in a sequential auction, the choice of winner at one node may affect the valuations at nodes higher in the tree. In particular, the equilibrium path may vary with different tie-breaking rules, leading to different prices, winners, and utilities.

As we will show in Sect. 4 there are a massive number of tie-breaking rules even in small sequential auctions. We emphasize, however, that our main result holds regardless of the tie-breaking rule: for *any* tie-breaking rule there is a sequential auction on which it induces a non-monotonic price trajectory. First, though, we will show that non-monotonicity occurs for perhaps the three most natural choices, namely **preferential-ordering**, **first-in-first-out** and **last-in-first-out**. Interestingly, these rules correspond to the fundamental data structures of priority queues, queues, and stacks in computer science.

Preferential Ordering (Priority Queue): In **preferential-ordering** each buyer is given a distinct rank. In case of a tie the buyer with the worst rank is eliminated. Without loss of generality, we may assume that the ranks correspond to a lexicographic ordering of the buyers. That is, the rank of a buyer is its index label and given a tie amongst all the buyers that wish to dropout of the auction we remove the buyer with the highest index. The preferential ordering tie-breaking rule corresponds to the data structure known as a *priority queue*.

First-In-First-Out (Queue): The **first-in-first-out** tie-breaking rule corresponds to the data structure known as a *queue*. The queue consists of those buyers in the auction that wish to dropout. Amongst these, the buyer at the front of the queue is removed. If multiple buyers request to be added to the queue simultaneously, they will be added lexicographically. Note though that this is different from preferential ordering as the entire queue will not, in general, be ordered lexicographically. For example, when at a fixed price p we remove the buyer i at the front of the queue this may cause new buyers to wish to dropout at price p , who will be placed behind the other buyers already in the queue.

Last-In-First-Out (Stack): The **last-in-first-out** tie-breaking rule corresponds to the data structure known as a *stack*. Again the stack consists of those buyers in the auction that wish to dropout. Amongst these, the buyer at the top of the stack (i.e. the back of the queue) is removed. If multiple buyers request to be added to the stack simultaneously, they will be added lexicographically. At first glance, this **last-in-first-out** rule appears more unusual than the previous two, but it still has a natural interpretation: it corresponds to settings where the buyer whose situation has changed most recently reacts the quickest.

In order to understand these tie-breaking rules it is useful to see how they apply on an example. In Fig. 3 the dropout vector is $(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5) = (40, 40, 40, 40, 40)$, but the three tie-breaking rules select three different winners.

On running the ascending price procedure, both Buyer 3 and Buyer 4 wish to drop out when the price reaches 40. In **preferential-ordering**, our choice set is then $\{3, 4\}$ and we remove the highest index buyer, namely Buyer 4. With the removal of Buyer 4, neither Buyer 1 nor Buyer 5 have an incentive to

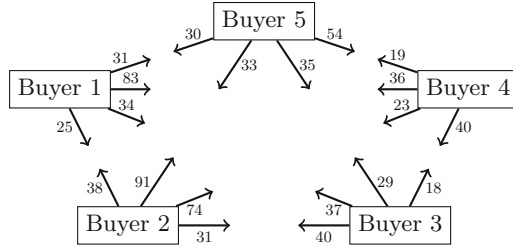


Fig. 3. An example to illustrate the three tie-breaking rules.

continue bidding so they both decide to dropout. Thus our choice set is now $\{1, 3, 5\}$ and **preferential-ordering** removes Buyer 5. With the removal of Buyer 5, now Buyer 2 no longer has an active participant it wishes to beat so the choice set is updated to $\{1, 2, 3\}$. The **preferential-ordering** rule now removes the buyers in the order Buyer 3, then Buyer 2 and lastly Buyer 1. Thus Buyer 1 wins under the **preferential-ordering** rule.

Now consider **first-in-first-out**. To allow for a consistent comparison between the three methods, we assume that when multiple buyers are simultaneously added to the queue they are added in decreasing lexicographical order. Thus our initial queue is $4 : 3$ and **first-in-first-out** removes Buyer 4 from the front of the queue. With the removal of Buyer 4, neither Buyer 1 nor Buyer 5 have an incentive to continue bidding so they are added to the back of the queue. Thus the queue is now $3 : 5 : 1$ and **first-in-first-out** removes Buyer 3 from the front of the queue. It then removes Buyer 5 from the front of the queue. With the removal of Buyer 5, we again have that Buyer 2 now wishes to dropout. Hence the queue is $1 : 2$ and **first-in-first-out** then removes Buyer 1 from the front of the queue. Thus Buyer 2 wins under the **first-in-first-out** rule.

Finally, consider the **last-in-first-out** rule. Again, to allow for a consistent comparison we assume that when multiple buyers are simultaneously added to the stack they are added in increasing lexicographical order. Thus our initial stack is $\frac{4}{3}$ and **last-in-first-out** removes Buyer 4 from the top of the stack. Again, Buyer 1 and Buyer 5 both now wish to drop out so our stack becomes $\frac{5}{1}$. Therefore Buyer 5 is next removed from the the top of the stack. At this point, Buyer 2 wishes to dropout so the stack becomes $\frac{2}{3}$. The **last-in-first-out** rule now removes the buyers in the order Buyer 2, then Buyer 1 and lastly Buyer 3. Thus Buyer 3 wins under the **last-in-first-out** rule.

We have now developed all the tools required to implement our sequential auction experiments. We describe these experiments and their results in Sect. 5. Before doing so, we present in Sect. 3 one sequential auction obtained via these experiments and verify that it leads to a non-monotonic price trajectory with each of the three tie-breaking rules discussed above. We then explain in Sect. 4 how to generalize this conclusion to apply to every tie-breaking rule.

3 An Auction with Non-monotonic Prices

Here we prove that the decreasing price anomaly is *not* guaranteed for sequential auctions with more than two buyers. Specifically, in Sect. 4 we prove the following result:

Theorem 5. *For any tie-breaking rule τ , there is a sequential auction on which it produces non-monotonic prices.*

In the rest of this section, we show that for all three of the tie-breaking rules discussed (namely, preferential-ordering, first-in-first-out and last-in-first-out) there is a sequential auction with with non-monotonic prices. Specifically, we exhibit a sequential auction with three buyers and eight items that exhibits non-monotonic prices.

Theorem 4. *There is a sequential auction which exhibits a non-monotonic price trajectory for the preferential-ordering, the first-in-first-out and the last-in-first-out rules.*

Proof. Our counter-example to the conjecture is a sequential auction with three buyers and eight identical items for sale. We present the first-price version where at equilibrium the buyers bid their dropout values in each time period; as discussed, the same example extends to second-price auctions. In our example, Buyer 1 has marginal valuations $\{55, 55, 55, 55, 30, 20, 0, 0\}$, Buyer 2 has marginal valuations $\{32, 20, 0, 0, 0, 0, 0, 0\}$, and Buyer 3 has marginal valuations $\{44, 44, 44, 44, 0, 0, 0, 0\}$. Let’s now compute the extensive forms of the auction under the three tie-breaking rules. We begin with the preferential-ordering rule. To compute its extensive form, observe that Buyer 1 is guaranteed to win at least two items in the auction because Buyer 2 and Buyer 3 together have positive value for six items. Therefore, the feasible set of sink nodes in the extensive form representation are shown in Fig. 4.



Fig. 4. Sink nodes of the extensive form game.

Given the valuations at the sink nodes we can work our way upwards recursively calculating the values at the other nodes in the extensive form representation. For example, consider the node $(x_1, x_2, x_3) = (4, 1, 2)$. This node has three children, namely $(5, 1, 2)$, $(4, 2, 2)$ and $(4, 1, 3)$; see Fig. 5(a). These induce a three-buyer auction as shown in Fig. 5(b). This can be solved using the ascending price procedure to find the dropout bids for each buyer. Thus we obtain that the value for the node $(x_1, x_2, x_3) = (4, 1, 2)$ is as shown in Fig. 5(c). Of course this node is particularly simple as, for the final round of the sequential auction, the corresponding auction with interdependent valuations is just a standard auction. That is, when the final

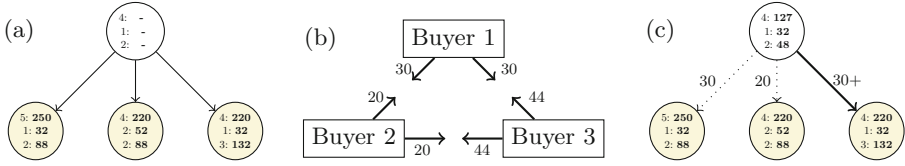


Fig. 5. Solving a subgame above the sinks.

item is sold, for any buyer i the value $v_{i,j}$ is independent of the buyer $j \neq i$. Nodes higher up the game tree correspond to more complex auctions with interdependent valuations. For example, the case of the source node $(x_1, x_2, x_3) = (0, 0, 0)$ is shown in Fig. 6. In this case, on applying the ascending price procedure, Buyer 1 is the first to dropout at price 15. At this point, both Buyer 2 and Buyer 3 no longer have a competitor that they wish to beat at this price, so they both want to dropout. With the **preferential-ordering** tie-breaking rule, Buyer 2 wins the item.

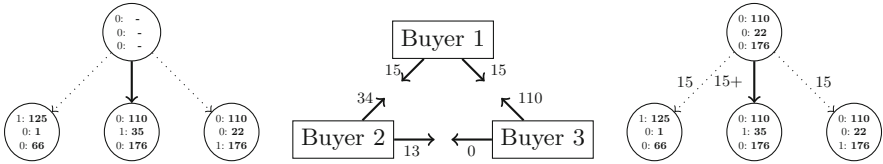


Fig. 6. Solving the subgame at the root.

Using similar arguments at each node verifies the concise extensive form representation of this example under the **preferential-ordering** tie-breaking rule. A figure showing the full extensive form tree is present in the full paper. The resultant price trajectory on the equilibrium path is $\{15, 17, 0, 0, 0, 0, 0\}$. That is, the price rises and then falls to zero – a non-monotonic price trajectory.

Exactly the same example works with the other two tie-breaking rules. The node values under **preferential-ordering** and **first-in-first-out** are the same, but these two rules do produce different winners at some nodes, for example the node $(3, 0, 2)$. In contrast, the **last-in-first-out** rule gives an extensive form where some nodes have different valuations than those produced by the other two rules. For example, for the node $(2, 0, 0)$ and its subgame the equilibrium paths and their prices differ. However, for all three rules the equilibrium path and price trajectory for the whole game is exactly the same. We remark that these observations will play a role when we prove that, for any tie-breaking rule, there is a sequential auction with non-monotonic prices. \square

Again, we emphasize that there is nothing inherently perverse about this example. The form of the valuation functions, namely decreasing marginal valuations, is standard. As explained, the equilibrium concept studied is the appropriate one for

sequential auctions. Finally, the non-monotonic price trajectory is not the artifact of an aberrant tie-breaking rule; we will now prove that non-monotonic prices are exhibited under any tie-breaking rule.

4 General Tie-Breaking Rules: Non-monotonic Prices

Next we prove that for any tie-breaking rule there is a sequential auction on which it produces a non-monotonic price trajectory. To do this, we must first formally define the set of all tie-breaking rules. Our definition will utilize the concept of an *overbidding graph*, introduced by Paes Leme et al. [20]. For any price p and any set of bidders S , the overbidding graph $G(S, p)$ contains a labelled vertex for each buyer in S and an arc (i, j) if and only if $v_{i,i} - p > v_{i,j}$. For example, recall the auction with interdependent valuations seen in Fig. 3. This is reproduced in Fig. 7 along with its overbidding graph $G(\{1, 2, 3, 4, 5\}, 40)$.

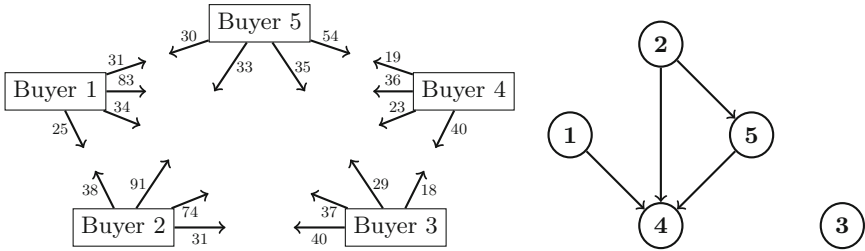


Fig. 7. The overbidding graph $G(\{1, 2, 3, 4, 5\}, 40)$.

But what does the overbidding graph have to do with tie-breaking rules? First, recall that the drop-out bid β_i is unique for any buyer i , regardless of the tie-breaking rule. Consequently, whilst the tie-breaking rule will also be used to order buyers that are eliminated at prices below the final price p^* , such choices are irrelevant with regards to the final winner. Thus, the only relevant factor is how a decision rule selects a winner from amongst those buyers S^* whose drop-out bids are p^* . Second, recall that a buyer *cannot* be eliminated if there remains another buyer still in the auction that it wishes to beat at price p^* . That is, buyer i must be eliminated after buyer j if there is an arc (i, j) in the overbidding graph. Thus, the order of eliminations given by the tie-breaking rule must be consistent with the overbidding graph. In particular, the winner can only be selected from amongst the *source vertices*² in the overbidding graph $G(S^*, p^*)$. For example, in Fig. 7 the source vertices are $\{1, 2, 3\}$. Note that this explains why the tie-breaking rules **preferential-ordering**, **first-in-first-out** and **last-in-first-out** chose Buyer 1, Buyer 2 and Buyer 3 as winners but none of them selected Buyer 4 or Buyer 5. Observe that the overbidding graph $G(S^*, p^*)$ is *acyclic*; if it contained

² A *source* is a vertex v with in-degree zero; that is, there no arcs pointing into v .

a directed cycle then the price in the ascending auction would be forced to rise further. Because every directed acyclic graph contains at least one source vertex, any tie-breaking rule does have at least one choice for winner. Thus a tie-breaking rule is simply a function $\tau : H \rightarrow \sigma(H)$, where the domain is the set of labelled, directed acyclic graphs and $\sigma(H)$ is the set of source nodes in H . Consequently, two tie-breaking rules are equivalent if they correspond to the same function τ . We are now ready to present our main result.

Theorem 5. *For any tie-breaking rule, there is a sequential auction with non-monotonic prices.*

We present here a sketch of our proof of this theorem; due to length restrictions the full proof is deferred. We consider the same example as in Theorem 4, and analyze the set of all possible tie-breaking rules in three-buyer auctions. We show that each tie-breaking rule produces an outcome from a set of exactly ten possible distinct extensive forms for this example. Of these ten classes, exactly five classes result in non-monotonicity. We then show that for any given tie-breaking rule from the other five classes it is possible to relabel the buyers in a way that the resulting equilibrium has a non-monotonic price trajectory.

5 Experiments

Our experiments were based on a dataset of over six million multi-buyer sequential auctions with non-increasing valuation functions randomly generated from different natural discrete probability distributions. Our goal was to observe the proportion of non-monotonic price trajectories and see how this varied with (i) the number of buyers, (ii) the number of items, (iii) the distribution of valuation functions, and (iv) the tie-breaking rule. For each auction we computed the subgame perfect equilibrium corresponding to the dropout bids and evaluated the prices on the equilibrium path to test for non-monotonicity. We repeated this test for each of the three tie breaking rules described in Sect. 2.4. The main conclusion to be drawn from these experiments is that non-monotonic prices are extremely rare. Of the 6,240,000 auctions, the **preferential-ordering**, **first-in-first-out** and **last-in-first-out** rules gave just 1,100, 986, and 1,334 violations of the declining price anomaly respectively. The overall observed rate of non-monotonicity over these 18 million tests was 0.000183. A detailed description of our dataset generation process and results are in the full paper.

References

1. Ashenfelter, O.: How auctions work for wine and art. *J. Econ. Perspect.* **3**(3), 23–36 (1989)
2. Ashta, A.: Wine auctions: more explanations for the declining price anomaly. *J. Wine Res.* **17**(1), 53–62 (2006)
3. Bae, J., Beigman, E., Berry, R., Honig, M., Vohra, R.: Sequential bandwidth and power auctions for distributed spectrum sharing. *J. Sel. Areas Commun.* **26**(7), 1193–1203 (2008)

4. Beggs, A., Graddy, K.: Declining values and the afternoon effect: evidence from art auctions. *Rand J. Econ.* **28**, 544–565 (1997)
5. van den Berg, G., van Ours, J., Pradhan, M.: The declining price anomaly in Dutch Dutch rose auctions. *Am. Econ. Rev.* **91**, 1055–1062 (2001)
6. Black, J., de Meza, D.: Systematic price differences between successive auctions are no anomaly. *J. Econ. Manag. Strategy* **1**(4), 607–628 (1992)
7. Buccola, S.: Price trends at livestock auctions. *Am. J. Agric. Econ.* **64**, 63–69 (1982)
8. Chanel, O., Gérard-Varet, L., Vincent, S.: Auction theory and practice: evidence from the market for jewellery. In: Ginsburgh, V., Menger, P. (eds.) *Economics of the Arts: Selected Essays*, pp. 135–149. North-Holland (1996)
9. Funk, P.: Auctions with interdependent valuations. *Int. J. Game Theory* **25**, 51–64 (1996)
10. Gale, I., Stegeman, M.: Sequential auctions of endogenously valued objects. *Games Econ. Behav.* **36**(1), 74–103 (2001)
11. Ginsburgh, V.: Absentee bidders and the declining price anomaly in wine auctions. *J. Polit. Econ.* **106**(6), 319–335 (1998)
12. Ginsburgh, V., van Ours, J.: On organizing a sequential auction: results from a natural experiment by Christie's. *Oxford Econ. Pap.* **59**(1), 1–15 (2007)
13. Hu, A., Zou, L.: Sequential auctions, price trends, and risk preferences. *J. Econ. Theory* **158**, 319–335 (2015)
14. Jehiel, P., Moldovanu, B.: Strategic nonparticipation. *Rand J. Econ.* **27**(1), 84–98 (1996)
15. Lambson, V., Thurston, N.: Sequential auctions: theory and evidence from the Seattle fur exchange. *Games Econ. Behav.* **37**(1), 70–80 (2006)
16. Lusht, K.: Order and price in a sequential auction. *J. Real Estate Finance Econ.* **8**, 259–266 (1994)
17. McAfee, P., Vincent, D.: The declining price anomaly. *J. Econ. Theory* **60**, 191–212 (1993)
18. Mezzetti, C.: Sequential auctions with informational externalities and aversion to price risk: decreasing and increasing price sequences. *Econ. J.* **121**(555), 990–1016 (2011)
19. Milgrom, P., Weber, R.: A theory of auctions and competitive bidding II. In: Klemperer, P. (ed.) *The Economic Theory of Auctions*. Edward Elgar (2000)
20. Paes Leme, R., Syrgkanis, V., Tardos, E.: Sequential auctions and externalities. In: *Proceedings of 23rd Symposium on Discrete Algorithms*, pp. 869–886. Society for Industrial and Applied Mathematics (2012)
21. Pesando, J., Shum, P.: Price anomalies at auction: evidence from the market for modern prints. In: Ginsburgh, V., Menger, P. (eds.) *Economics of the Arts: Selected Essays*, pp. 113–134. North-Holland (1996)
22. Rodriguez, G.: Sequential auctions with multi-unit demands. *B.E. J. Theor. Econ.* **9**(1), 1–35 (2009)
23. Sosnick, S.: Bidding strategy at ordinary auctions. *J. Farm Econ.* **45**(1), 8–37 (1961)
24. Thiel, S., Petry, G.: Bidding behaviour in second-price auctions: rare stamp sales, 1923–1937. *Appl. Econ.* **27**(1), 11–16 (1995)
25. Tu, Z.: A resale explanation for the declining price anomaly in sequential auctions. *Rev. Appl. Econ.* **6**(1–2), 113–127 (2010)
26. Weber, R.: Multiple object auctions. In: Engelbrecht-Wiggans, R., Shubik, M., Stark, R. (eds.) *Auctions, Bidding and Contracting: Use and Theory*, pp. 165–191. New York University Press (1983)